MATHEMATICS

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Abstract

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ESTIMATES OF THE GREEN FUNCTION OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE

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1. Let Ω be an open bounded domain of n-dimensional space, bounded by a closed surface S of class $A^{(1,\lambda)}$ (^{1}). Let the functions $a_{ik}(x)$ $(i, k = 1, 2, \ldots, n)$, $a(x)$ $(x = (x_1, x_2, \ldots, x_n))$ be defined on $\overline{\Omega}$. Suppose that the functions $\partial a_{ik}(x)/\partial x_n$, $a(x)$ are continuous on $\overline{\Omega}$ and satisfy a Hölder condition at every interior point of Ω. Let the function $\sigma(y)$ be defined and continuous on S. Suppose

$$a_{ik}(x) = a_{ki}(x), \quad \sum_{i,k=1}^{n} \gamma_i \gamma_k a_{ik}(x) \geq \lambda_0 \sum_{i=1}^{n} \gamma_i^2$$

for any $\gamma_1, \ldots, \gamma_n$ and some $\lambda_0 > 0$, $a(x) \geq a_0 > 0$, and $\sigma(y) \geq 0$.

Consider the boundary-value problem

$$\frac{\partial v}{\partial t} - \sum_{i,k=1}^{n} \frac{\partial}{\partial x_i} \left[a_{ik}(x) \frac{\partial v}{\partial x_k} \right] + a(x) v = 0 \quad (t > 0,\ x \in \Omega); \tag{1}$$

$$\lim_{\substack{x \to y \\ x \in N_y}} \sum_{i,k=1}^{n} a_{ik}(x) \frac{\partial v}{\partial x_k} \cos(N_y, x_i) + \sigma(y) v = 0 \quad (t > 0,\ y \in S); \tag{2}$$

$$v(0, x) = v_0(x) \quad (x \in \overline{\Omega}), \tag{3}$$

where N_y is the vector of the outward normal to the surface S at the point y.

In (^{2}) it is shown that, under the conditions formulated above and for any function $v_0(x)$ continuous on $\overline{\Omega}$, this problem has a unique solution $v(t, x)$, continuous in the cylinder $[0, \infty) \times \overline{\Omega}$ and continuously differentiable once with respect to t and twice with respect to x in the cylinder $(0, \infty) \times \Omega$. Problem

(1)–(2)–(3) is reduced to a Volterra-type integral equation with weak singularities, and therefore its solution $v(t, x)$ can be found by the method of successive approximations.

It turns out that the solution $v(t, x)$ can be represented in the form

$$v(t, x) = \int_{\Omega} G(t; x, y) v_0(y)\, dy. \tag{4}$$

The function $G(t; x, y)$ is called the Green function. It is continuous in the cylinder $(0, \infty) \times \overline{\Omega} \times \overline{\Omega}$, continuously differentiable once with respect to t and twice with respect to x in the cylinder $(0, \infty) \times \Omega \times \Omega$, and satisfies equation (1) and the boundary condition (2). Further, the function $G(t; x, y)$ is nonnegative and symmetric with respect to x and y. Finally, for any $t, \tau > 0$ the identity

$$G(t + \tau; x, y) = \int_{\Omega_z} G(t; x, z) G(\tau; z, y)\, dz \tag{5}$$

holds.

The properties of $G(t;\, x, y)$ formulated here (possibly under more stringent restrictions) were apparently known earlier. However, the proof of these properties, as well as of the theorems formulated below, under the weakest possible assumptions is important in the investigation of nonlinear equations.

2. **Theorem 1.** *Let* $t_0 > 0$*. If* $t \in (0, t_0]$*, then*

$$0 \leqslant G(t;\, x, y) \leqslant C(t_0) \exp\left\{ -\frac{\delta(t_0)\, r_{xy}^2}{t} \right\} t^{-n/2}. \tag{6}$$

If $t \geqslant t_0$*, then*

$$0 \leqslant G(t;\, x, y) \leqslant C(t_0) \exp\{-a_0 t\}; \tag{7}$$

here $C(t_0)$*,* $\delta(t_0) > 0$*.*

Theorem 2. *Let* $\nu_0 \in (0, 1)$*,* $\varepsilon_0 \in (0, n/2)$*,* $\nu \in [0, \nu_0]$*,* $\varepsilon \in (0, \varepsilon_0]$*. If* $t \in (0, t_0]$*, then*

$$r_{xy}^{-\nu} \left| G(t;\, x, z) - G(t;\, y, z) \right| \leqslant$$

$$\leqslant C(t_0, \nu_0, \varepsilon) \exp\left\{ -\frac{\delta(t_0, \nu_0, \varepsilon)\, r^2}{t} \right\} \cdot \min\left\{ t^{-\frac{n+\nu}{2} + \varepsilon} r^{-2\varepsilon},\; t^{-\frac{n+\nu}{2} - \varepsilon} \right\}, \tag{8}$$

where $r = \min\{r_{xz}, r_{yz}\}$*. If* $t \geqslant t_0$*, then*

$$r_{xy}^{-\nu} \left| G(t;\, x, z) - G(t;\, y, z) \right| \leqslant C(t_0, \nu_0) \exp\{-a_0 t\}. \tag{9}$$

Using the symmetry of the function $G(t;\, x, y)$ and identity (5), we can estimate the Hölder coefficients of this function with respect to the totality of the variables x and y.

3. **Theorem 3.** *Let the function* $\sigma(y)$ *satisfy the condition*

$$|\sigma(y_1) - \sigma(y_2)| \leqslant C|y_1 - y_2|^h \quad (C > 0,\ 0 < h < 1). \tag{10}$$

If $t \in (0, t_0]$, *then*

$$\left|\frac{\partial}{\partial x_i} G(t;\, x, y)\right| \leqslant$$

$$\leqslant C(t_0, \varepsilon_0) \exp\left\{-\frac{\delta(t_0, \varepsilon)\, r_{xy}^2}{t}\right\} \cdot \min\left\{t^{-\frac{n+1}{2}+\varepsilon} r_{xy}^{-2\varepsilon},\ t^{-\frac{n+1}{2}-\varepsilon}\right\}. \tag{11}$$

If $t \geqslant t_0$, *then*

$$\left|\frac{\partial}{\partial x_i} G(t;\, x, y)\right| \leqslant C(t_0) \exp\{-a_0 t\}. \tag{12}$$

Identity (5) makes it possible to estimate the derivatives of $G(t;\, x, y)$ with respect to the totality of the variables x and y. If $t \in (0, t_0]$, then

$$\left|\frac{\partial^2}{\partial x_i \partial y_k} G(t;\, x, y)\right| \leqslant$$

$$\leqslant C(t_0, \varepsilon) \exp\left\{-\frac{\delta(t_0, \varepsilon)}{t} r_{xy}^2\right\} \cdot \min\left\{t^{-\frac{n+2}{2}+\varepsilon} r_{xy}^{-2\varepsilon},\ t^{-\frac{n+2}{2}-\varepsilon}\right\}. \tag{13}$$

If $t \geqslant t_0$, then

$$\left|\frac{\partial^2}{\partial x_i \partial y_k} G(t;\, x, y)\right| \leqslant C(t_0) \exp\{-a_0 t\}. \tag{14}$$

In the general case, when the function $\sigma(y)$ is only continuous, the function $G(t;\, x, y)$ can be represented in the form of a sum of two terms G_1 and G_2,

for the first of which estimate (11) is valid, and for the second, for $t \in (0, t_0]$, the estimate

$$\left\|\int_{\Omega_y} \frac{\partial}{\partial x_i} G(t; x, y) f(y)\, dy\right\|_{L_{p_1}(\Omega)} \leqslant C(t_0, \beta) \|f(x)\|_{L_p(\Omega)} \tag{15}$$

for any values

$$\alpha \in \left(\frac{1}{2}, \frac{n+1}{2}\right), \quad p \in \left(1, \frac{n}{2\alpha - 1}\right), \quad p_1 = \frac{np}{n - (2\alpha - 1)p},$$

$\beta \in \left(\alpha + \frac{1-\lambda}{2}, \alpha + 1\right)$, and $f(x) \in L_p(\Omega)$.

Theorem 4. Let (10) be satisfied. Let $\nu_0 \in (0, \min\{\lambda, h\})$ and $\nu \in [0, \nu_0]$. If $t \in (0, t_0]$, then

$$r_{xy}^{-\nu} \left|\frac{\partial}{\partial x_i} G(t; x, z) - \frac{\partial}{\partial y_i} G(t; y, z)\right| \leqslant$$

$$\leqslant C(t_0,\nu_0,\varepsilon)\exp\left\{-\frac{\delta(t_0,\nu_0,\varepsilon)r^2}{t}\right\}\cdot\min\left\{t^{-\frac{n+1+\nu}{2}+\varepsilon}r^{-2\varepsilon},t^{-\frac{n+1+\nu}{2}-\varepsilon}\right\},\qquad(16)$$

where $r=\min\{r_{xz},r_{yz}\}$. If $t\geqslant t_0$, then

$$r_{xy}^{-\nu}\left|\frac{\partial}{\partial x_i}G(t;x,z)-\frac{\partial}{\partial y_i}G(t;y,z)\right|\leqslant C(t_0,\nu_0)\exp\{-a_0t\}.\qquad(17)$$

Identity (15) makes it possible to estimate the Hölder coefficients of the functions

$$\frac{\partial}{\partial x_i}G(t;x,y),\qquad\frac{\partial^2}{\partial x_i\partial y_k}G(t;x,y)$$

with respect to the totality of the variables x and y.

4. Since $S\in A^{(1,\lambda)}$, there exists a number $r_0>0$ such that for any point $x\in\Omega$ at distance from S no greater than r_0, there is a unique nearest point $p_x\in S$. Obviously, $x\in N_{p_x}$. For such x we put

$$\frac{dv}{dT_{p_x}}=\sum_{i,k=1}^{n}a_{ik}(x)\frac{\partial v}{\partial x_k}\cos(N_{p_x},x_i).\qquad(18)$$

Theorem 5. If $t\in(0,t_0]$, then

$$\left|\frac{d}{dT_{p_x}}G(t;x,y)\right|\leqslant$$

$$\leqslant C(t_0,\varepsilon)\exp\left\{-\frac{\delta(t_0,\varepsilon)r_{xy}^2}{t}\right\}\cdot\min\left\{t^{-\frac{n+1}{2}+\varepsilon}r_{xy}^{-2\varepsilon},t^{-\frac{n+1}{2}-\varepsilon}\right\}.\qquad(19)$$

If $t\geqslant t_0$, then

$$\left|\frac{d}{dT_{p_x}}G(t;x,y)\right|\leqslant C(t_0)\exp\{-a_0t\}.\qquad(20)$$

Theorem 6. Let $\nu_0\in(0,\lambda)$, $\nu\in[0,\nu_0]$. If $t\in(0,t_0]$, then

$$r_{xy}^{-\nu}\left|\frac{\partial}{\partial T_{p_x}}G(t;x,z)-\frac{\partial}{\partial T_{p_y}}G(t;y,z)\right|\leqslant$$

$$\leqslant C(t_0,\nu_0,\varepsilon)\exp\left\{-\frac{\delta(t_0,\nu_0,\varepsilon)r^2}{t}\right\}\cdot\min\left\{t^{-\frac{n+1+\nu}{2}+\varepsilon}r^{-2\varepsilon},t^{-\frac{n+1+\nu}{2}-\varepsilon}\right\},\qquad(21)$$

where $r = \min\{r_{xz}, r_{yz}\}$. If $t \geqslant 0$, then

$$r_{xy}^{-\nu} \left| \frac{\partial}{\partial T_{p_x}} G(t; x, z) - \frac{d}{dT_{p_y}} G(t; y, z) \right| \leqslant C(t_0, \nu_0) \exp\{-a_0 t\}. \tag{22}$$

Theorem 7. Let $x, y, z \in S$ and $t \in (0, t_0]$. Then

$$\left| \frac{d}{dT_x} G(t; x, y) \right| \leqslant$$

$$\leqslant C(t_0, \varepsilon) \exp\left\{ -\frac{\delta(t_0, \varepsilon) r_{xy}^2}{t} \right\} \cdot \min\left\{ t^{-\frac{n+1-\lambda}{2}+\varepsilon} r_{xy}^{-2\varepsilon}, t^{-\frac{n+1-\lambda}{2}-\varepsilon} \right\}; \tag{23}$$

$$r_{xy}^{-\nu} \left| \frac{d}{dT_x} G(t; x, z) - \frac{d}{dT_y} G(t; y, z) \right| \leqslant \tag{24}$$

$$\leqslant C(t_0, \nu_0, \varepsilon) \exp\left\{ -\frac{\delta(t_0, \nu_0, \varepsilon) r^2}{t} \right\} \cdot \min\left\{ t^{-\frac{n+1-\lambda+\nu}{2}+\varepsilon} r_{xy}^{-2\varepsilon}, t^{-\frac{n+1-\lambda+\nu}{2}-\varepsilon} \right\},$$

where $r = \min\{r_{xz}, r_{yz}\}$.

5. The same estimates hold for the Green function $\overline{G}(t; x, y)$ of the first boundary-value problem. We also note the inequality

$$0 \leqslant \overline{G}(t; x, y) \leqslant G(t; x, y). \tag{25}$$

6. The estimates established in this paper find application in the investigation of fractional powers of elliptic operators.

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Note: Figure translations are in progress. See original paper for figures.

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