AN ANALOGUE OF SCHMIDT' S METHOD FOR A NONLINEAR EQUATION

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Abstract

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AN ANALOGUE OF SCHMIDT' S METHOD FOR A NONLINEAR EQUATION

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Schmidt' s method ([1]) makes it possible to reduce the solution of linear integral equations to the solution of such equations with a fixed degenerate kernel. One can propose a certain analogue of this method for nonlinear equations as well. Thus the questions of existence and uniqueness of the solution of a nonlinear problem are reduced to the investigation of a certain system of finite equations. The possibility of applying this method is connected with obtaining certain a priori estimates.

In the present note we shall show this on the example of the equation

$$u(P) = \iint_D K(P,Q)u^2(Q)\,dQ + \psi(P), \tag{1}$$

which arises in the boundary-value problem

$$\Delta u + u^2 = 0, \qquad u\big|_\Gamma = f(s), \tag{2}$$

where Γ is the boundary of a two-dimensional bounded domain D, for which there exists the Green' s function $K(P,Q)$ of the Dirichlet problem. Problem (2) is considered in the class of functions $C^{(2)}$. The only condition imposed on the function $f(s)$ is continuity.

It will be proved that, for the given domain D and function $f(s)$, this problem reduces to a certain nonlinear integral equation with a fixed degenerate kernel.

Lemma 1. Let a function $v(P)$ be given which has continuous first derivatives in $D+\Gamma$ and continuous second derivatives in D, is everywhere positive in D, and satisfies $v|_\Gamma = 0$. Then, for any solution $U(P)$ of the boundary-value problem (2), the inequality

$$\iint_D v(Q)U^2(Q)\,dQ \le A_v^2, \tag{3}$$

holds, where

$$A_v = \frac{1}{2}\left[\iint_D \frac{(\Delta v(Q))^2}{v(Q)}\,dQ\right]^{1/2} + \frac{1}{2}\left[\iint_D \frac{(\Delta v(Q))^2}{v(Q)}\,dQ + 4\int_\Gamma |f v_\nu|\,ds\right]^{1/2}$$

and v_ν is the derivative of the function v in the direction of the outward normal ν.

This estimate is obtained from Green' s formula for the functions $v(P)$ and the solution $U(P)$ of the boundary-value problem (2). Estimate (3) can also be obtained directly from equation (1).

It is clear that there exist functions $v(P)$ for which the quantity A_v is finite. For example, as the function $v(P)$ one may take the first eigenfunction of the boundary-value problem

$$\Delta\varphi + \lambda\varphi = 0, \qquad \varphi\big|_\Gamma = 0. \tag{4}$$

Lemma 2. For any solution $U(P)$ of the integral equation (1), the estimate

$$\max_D |U(P)| \le C_v + a, \tag{5}$$

holds.

where C_ν is the positive root of the equation:

$$x^3 = B_\nu (x+a)^2,$$

$$B_\nu = A_\nu^4 \max_D \iint_D \frac{K^3(P,Q)}{\nu^2(Q)}\,dQ, \qquad a = \max_\Gamma |f(s)|.$$

Proof. From the integral equation (1) and the estimate (3) it follows that

$$U(P) - \psi(P) \le A_\nu \left[\iint_D \frac{K^2(P,Q)U^2(Q)}{\nu(Q)}\,dQ\right]^{1/2}.$$

Now, using the integral equation (1) once more, we obtain:

$$\max_D (U(P) - \psi(P)) \le C_\nu;$$

whence the required estimate (5) follows.

Let us now consider the question of finding all solutions of the boundary-value problem (2), which is equivalent to finding all solutions of the integral equation (1).

Represent the kernel $K(P,Q)$ of the integral equation (1) in the form of the sum of two kernels:

$$K(P,Q) = M_n(P,Q) + \Gamma_n(P,Q), \tag{6}$$

where

$$M_n(P,Q) = \sum_{k=1}^{n} \frac{\Phi_k(P)\Phi_k(Q)}{\lambda_k},$$

and $\Phi_k(P)$, λ_k are the eigenfunctions and eigenvalues of the boundary-value problem (4).

Put

$$\gamma_n = \max_D \iint_D |\Gamma_n(P,Q)|\,dQ.$$

Theorem. There exist a number n and a nonlinear operator $R[V(P)]$ such that all solutions $U(P)$ of the integral equation (1) are representable in the form

$$U(P) = V(P) + R[V(P)],$$

where $V(P)$ is a solution of the integral equation with fixed degenerate kernel $M_n(P,Q)$:

$$V(P) = \iint_D M_n(P,Q)(V(Q) + R[V(Q)])^2\,dQ, \tag{7}$$

satisfying the condition

$$\gamma_n \max_D |V(P)| < 1/4. \tag{8}$$

Proof. Equation (1) is equivalent to the system of equations

$$V(P) = \iint_D M_n(P,Q)(V(Q) + W(Q))^2\,dQ + \psi(P); \tag{9}$$

$$W(P) = \iint_D \Gamma_n(P,Q)(V(Q) + W(Q))^2\,dQ. \tag{10}$$

Consider separately equation (10) for fixed n. Suppose a continuous function $V(Q)$ is given, satisfying condition (8). Then equation (10) has a continuous solution $W(P)$, which is found by the method of successive Picard approximations according to the scheme

$$W(P) = \lim_{k\to\infty} W_k(P),$$

where

$$W_k(P) = \iint_D \Gamma_n(P,Q)(V(Q) + W_{k-1}(Q))^2\, dQ \quad (k = 2, 3, ...), \tag{11}$$

$$W_1(P) = \iint_D \Gamma_n(P,Q)V^2(Q)\, dQ.$$

In this case the solution $W(P)$ satisfies the inequality

$$|W(P)| \leq \frac{1}{2\gamma_n} q - b, \quad \text{where } q = 1 - \sqrt{1 - 4b\gamma_n}, \quad b = \max_D |V(P)|.$$

The maximum deviation of the function $W_k(P)$, obtained at the k-th step in solving equation (11), from the exact solution $W(P)$ does not exceed the quantity

$$|W_k(P) - W(P)| \leq \frac{1}{\gamma_n} q^{k-1} \left(\frac{q}{2} - \gamma_n b\right)(q - \gamma_n b) \quad (k = 1, 2, 3, ...).$$

Consequently, under condition (8), equation (10) admits a representation of the solution $W(P)$ in terms of the function $V(P)$ in the form $W(P) = R[V(P)]$, where the operator $R[V]$ is given by the formulas

$$R[V(P)] = \lim_{k\to\infty} R_k[V(P)],$$

$$R_k[V(P)] = \iint_D \Gamma_n(P,Q_1)\left(V(Q_1) + \iint_D \Gamma_n(Q_1,Q_2)\,(V(Q_2) + \cdots \right.$$

$$\left. \cdots + \left(\iint_D \Gamma_n(Q_{k-1},Q_k) V^2(Q_k)\, dQ_k\right)^2 \cdots\right)^2 \Bigg)\, dQ_1 \quad (k = 1, 2, ...).$$

In the class of functions $W(P)$ satisfying the condition

$$2\gamma_n \max_D |V(P) + W(P)| < 1, \tag{12}$$

the solution of equation (10) is unique.

Let us now consider jointly the system of equations (9), (10). For any two functions $V(P)$ and $W(P)$ satisfying this system, the a priori estimates

$$\max_D |V(P)| \leq M_1, \qquad \max_D |V(P) + M(P)| \leq M_2 \tag{13}$$

hold.

For example, by virtue of Lemma 2, as M_1 and M_2 one may take the numbers

$$M_1 = (C_v + a)^2 \max_D \left[S \iint_D K^2(P,Q)\, dQ_1 \right]^{-1/2} + a, \qquad M_2 = C_v + a,$$

where S is the area of the domain D.

By virtue of the a priori estimates (13), the assertion of the theorem follows from the fact that $\gamma_n \to 0$ as $n \to \infty$, and is valid for every $n \geq n_0$, where $n_0 = \max\{n_1, n_2\}$, and n_1 and n_2 are chosen from the conditions $4\gamma_{n_1} M_1 < 1$, $2\gamma_{n_2} M_2 < 1$.

Corollary. The number of solutions of the boundary-value problem (2) is determined by the number of solutions of equation (7) satisfying condition (8).

Remark. The operator $R[V]$ depends on the number n, and

$$\max_D |R[V(P)]| \to 0$$

as $n \to \infty$ for a fixed function $V(P)$.

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References

1. E. Schmidt, *Math. Ann.*, **64**, 161 (1907).

Note: Figure translations are in progress. See original paper for figures.