SOME ESTIMATES OF ANALYTIC CAPACITY

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Abstract

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Let F be a bounded closed set of points of the complex plane R; let B_1 be the set of all functions f, regular in $R \setminus F$ and such that $f(\infty) = 0$, $\sup_{z\in R/F} |f(z)| \leqslant 1$.

The analytic capacity of F is the number

$$\Omega(F) = \sup_{f\in B_1} \left| \lim_{z\to\infty} zf(z) \right| \tag{1}$$

(for this concept see ([1–7])). In this note some estimates for $\Omega(F)$ will be established.

Let G_∞ be that one of the connected components of $R \setminus F$ which contains ∞; let $G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots$ be a sequence of domains exhausting G_∞, where the boundary ∂G_n of the domain G_n consists of a finite number of pairwise nonintersecting rectifiable closed Jordan curves; let $|\partial G_n|$ denote the length of ∂G_n. Put $l(F) = \inf_{\{G_n\}} \lim_{n\to\infty} |\partial G_n|$ (the infimum is taken over all sequences $\{G_n\}_{n=1}^{\infty}$ of the indicated kind).

Fix some neighborhood g of the set F. Let φ be a function given on F. Taking $\varepsilon > 0$, consider the collection $\mathfrak{R}(\varepsilon, \varphi)$ of all fractions r of the form

$$r(z) = \sum_{k=1}^{n} \frac{\lambda_k}{z - a_k} \quad (a_1, a_2, \dots, a_n \in g \setminus F)$$

such that $\sup_{z\in F} |\varphi(z) - r(z)| < \varepsilon$ (it is not excluded here that $\mathfrak{R}(\varepsilon, \varphi) = \Lambda$). Let

$$p_F(\varepsilon, \varphi) = \inf_{r\in\mathfrak{R}(\varepsilon,\varphi)} \left\{ \sum |\lambda_k| \right\}$$

(if $\mathfrak{R}(\varepsilon, \varphi) = \Lambda$, we set $p_F(\varepsilon, \varphi) = +\infty$), and

$$p_F(\varphi) = \lim_{\varepsilon\to 0} p_F(\varepsilon, \varphi).$$

It is easy to see that $p_F(\varphi)$ does not depend on the choice of the neighborhood g. It is known that if $l(F) < +\infty$, then

$$\Omega(F) = p_F(1) \tag{2}$$

$(1(z) \equiv 1)$ ([6,7]).

Using (2), it is easy to prove the following theorem:

Theorem 1. Let F be a bounded closed set; $l(F) < +\infty$; let W be a continuous function given on F. Suppose that also $l(W(F)) < +\infty$. Let v be some neighborhood of $W(F)$. Then

$$\Omega(F) \leqslant \left[\frac{l(F)}{2\pi}\varepsilon + \sup_{a \in v \setminus W(F)} p_F(\varepsilon, (W-a)^{-1})\right] \Omega(W(F)). \tag{3}$$

Here

$$(W-a)^{-1}(\zeta) = \frac{1}{W(\zeta)-a},$$

and ε is any positive number.

From inequality (3) it follows that

$$\Omega(F) \leqslant \sup_{a \in v \setminus W(F)} p_F((W-a)^{-1})\, \Omega(W(F))^{*}. \tag{4}$$

* We shall assume that $+\infty \cdot 0 = +\infty$.

We note that inequality (4) becomes an equality when W is a linear mapping.

Example. Let F be the union of n rectilinear segments $[e^{2\pi ki/n}, re^{2\pi ki/n}]$ $(k = 0, 1, \dots, n-1;\ r > 1)$.

The function $W(z) = z^n$ maps F onto the segment of the real axis $W(F) = [1, r^n]$. At the same time it is easy to estimate $p_F((W-a)^{-1})$, if a is near $W(F)$:

$$\frac{1}{W(z)-a} = \sum_{k=0}^{n-1} \frac{1}{n\left(|a|^{1/n}e^{2\pi ki/n}\right)^{n-1}} \frac{1}{z-|a|^{1/n}e^{2\pi ki/n}},$$

$$p_F((W-a)^{-1}) \leq |a|^{(1-n)/n}.$$

Since $\Omega(W(F)) = \dfrac{r^n-1}{4}$ ([1]), it follows that

$$\Omega(F) \le \frac{r^n - 1}{4}.$$

Estimates (3) and (4) can be applied in the following direction. Let γ be a Jordan arc given by the equation $w = W(t)$, $t \in [0,1]$, and let F be a closed subset of the segment $[0,1]$. It is known that $\operatorname{mes} F/4 \le \Omega(F) \le \operatorname{mes} F/\pi$ ([1]). Therefore inequalities (3) and (4) make it possible in some cases to obtain effective estimates for the analytic capacity of sets lying on Jordan arcs. Such estimates are quite easy to obtain, for example, in the case when the arc γ is smooth, and $W'(t)$ satisfies the Lipschitz condition:

$$|W'(t_1) - W'(t_2)| \le \text{const.}\, |t_1 - t_2| \qquad (t_1, t_2 \in [0,1]).$$

Let γ be a closed rectifiable Jordan curve; G the interior of γ; W a function realizing a conformal mapping of the unit disk onto G. Then the curve γ can be given by the equation $w = W(e^{i\varphi})$ $(\varphi \in [0, 2\pi])$.

Using estimate (3), we obtain the following theorem:

Theorem 2. *Let F be a closed set lying on the unit circle, and suppose that for all $r \in [r_0, 1)$, $0 < r_0 < 1$,*

$$|W'(re^{i\varphi})| \ge \delta > 0$$

$(e^{i\varphi} \in F)$. Then

$$\Omega(F) \le \left\{ \frac{1}{\delta} + \sup_{\substack{z_0 \in \nu \\ r_0 < r < 1}} \frac{1}{\pi} \int_F \left| \frac{1}{W(r\xi) - W(z_0)} - \frac{1}{W'(z_0)(r\xi - z_0)} \right| |d\xi| \right\} \Omega(W'(F)). \tag{5}$$

Here ν is the intersection of any fixed arbitrarily small neighborhood of F with the unit disk.

In ([1]) it was established that $\Omega(P) > 0$ if P is a set of positive length lying on an analytic curve. The corollary formulated below, as well as Theorems 3 and 4, allow this result to be extended to a broader class of sets.

Corollary. If the expression standing in braces in (5) is finite and the length of $W(F)$ is positive, then the analytic capacity of $W(F)$ is also positive.

Indeed, if the length of $W(F)$ is positive, then the length of F is also positive ([8]), and hence $\Omega(F) > 0$ ([1]); our assertion now follows from estimate (5).

In particular, if

$$\sup_{\substack{|z_0|<1\\0<r<1}} \int_{|\xi|=r} \left| \frac{1}{W(\xi)-W(z_0)} - \frac{1}{W'(z_0)(\xi-z_0)} \right| |d\xi| < +\infty, \tag{6}$$

then the capacity of a set lying on γ is positive if and only if its length is positive.

We note that fulfillment of the inequality

$$\inf_{\substack{1>r>r_0\\e^{i\varphi}\in F}} |W'(re^{i\varphi})| > 0$$

can always be ···

to obtain, by narrowing F somewhat and using the theorems of Luzin, Egorov, and the known properties of functions of class H_1, to which W' belongs ([8]).

It is easy to show, using a theorem of Kellogg ([9]), that condition (6) is satisfied, for example, when the curve γ is smooth, and the angle $\theta(s)$ of inclination of the tangent to the real axis, as a function of the arc length s, satisfies a Lipschitz condition:

$$|\theta(s') - \theta(s'')| \leqslant \text{const} \cdot |s'-s''|^{\alpha}, \qquad 0 < \alpha \leqslant 1.$$

Somewhat more general conditions, formulated in terms of the moduli of continuity of W' or $\theta(s)$ and sufficient for the finiteness of the right-hand side in (5), can be obtained using results of S. Ya. Al' per ([11])*.

In the proof of the following theorem an important role is played by the relation, established in ([5]), between the analytic capacity (1) and

$$\sup_{f\in E_p^1(\{G_n\})} \left| \lim_{z\to\infty} zf(z) \right|.$$

Theorem 3. Let F be a closed set lying on the unit circle; let the function W map the unit disk onto the interior of a closed rectifiable Jordan curve γ.

a) Suppose that there exist numbers $p > 1$ and $A > 0$ such that for every measurable function α, defined on γ, with $\sup_{w\in\gamma} |\alpha(w)| \leqslant 1$, the inequality

$$\sup_{0<r<1} \int_{W(rF)} |u_\alpha(\zeta)|^p \, |d\zeta| \leqslant A,$$

holds, where

$$u_\alpha(\zeta) = \int_\gamma \frac{\alpha(\tau) Z'(\tau)\, d\tau}{\tau - \zeta};$$

$Z(w)$ is the function inverse to $W(z)$.

b) Suppose that for all $r \in [r_0, 1)$ $(0 < r_0 < 1)$

$$|W'(re^{i\varphi})| \geqslant \delta > 0 \quad (e^{i\varphi} \in F).$$

Then

$$\Omega(F) \leqslant \frac{1}{\delta} \left\{ \sqrt[p]{A} + [\operatorname{mes} W(F)]^{1/q} \right\} (2\pi)^{-1/q} [\Omega(W(F))]^{1/p}, \qquad q = \frac{p}{p-1}.$$

Thus, if the length $W(F)$ is positive and the hypotheses of Theorem 3 are satisfied, then $\Omega(W(F)) > 0$.

It follows from Theorem 3 that, if the curve γ is smooth and

$$\int_\gamma \frac{\varphi(t)\, dt}{t - z} \qquad (z \in G)$$

represents a function of class $E_p(G)$ ([6]) whenever $\varphi \in L_p(\gamma)$ $(p > 1)$, then the length and capacity of any set $e \subset \gamma$ are simultaneously positive or not.

Applying a well-known theorem of Riesz (([10]), p. 149), one can indicate simple conditions sufficient for the set $W(F)$, lying on γ, to have the property described in the statement of Theorem 3.

Theorem 4. Let $p > 1$,

$$\sup_{0<r<1} \int_{|t|=1} \left(\int_{|z|=r} \left| \frac{1}{W(z) - W(t)} - \frac{1}{W'(z)(z-t)} \right|^p |dz| \right) |dt| < +\infty, \quad (7)$$

$$\sup_{\substack{r_0<r<1 \\ e^{i\varphi}\in F}} |W'(re^{i\varphi})| < +\infty, \qquad \inf_{\substack{r_0<r<1 \\ e^{i\varphi}\in F}} |W'(re^{i\varphi})| > 0.$$

Then condition a) of Theorem 3 is satisfied.

In particular, if (7) is satisfied, then the analytic capacity of every closed subset of γ of positive length is positive.

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* At the Fifth All-Union Conference on the Theory of Functions, L. D. Ivanov formulated a theorem establishing that $\Omega(F) > 0$ if F lies on a smooth curve subject to certain additional conditions and $\operatorname{mes} F > 0$.

As in the case of Theorem 2, it is easy to indicate simple conditions of a geometric nature that ensure the fulfillment of (7) (see, in particular, Lemma 1 of the paper [12]).

We do not know whether a set P, lying on a rectifiable (or even smooth) Jordan curve, can have positive length and zero analytic capacity.

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Note: Figure translations are in progress. See original paper for figures.