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Abstract

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MATHEMATICS

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On the Theory of Discontinuous Variational Problems with Movable Endpoints in Space

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The theory of variational problems with extremals having corner points or discontinuities of the first kind finds many applications in technology (¹). The present note is devoted to finding necessary minimum conditions for discontinuous variational problems with movable endpoints in a space of many dimensions.

In contrast to other types of problems, which are essentially continuous but have broken or discontinuous extremals, here the problem itself is "discontinuous." Discontinuous problems arise in the study of wave processes in media with jumpwise-varying properties (²). The theory of plane discontinuous problems in nonparametric form was developed in (³).

1. Let $(x, y_1, y_2, \dots, y_n) \equiv (x, y)$ be a point of the $(n+1)$-dimensional Euclidean space E^{n+1}. In E^{n+1} let a domain R with boundary S be given. Suppose that inside R there is a certain broken curve E_{102}, consisting of arcs E_{10} and E_{02}. The curve E_{102} is represented by the equations

$$y_i = y_i(x) \begin{cases} \bar{y}_i(x), & x^1 \le x \le x^0, \\ \overset{+}{y}_i(x), & x^0 \le x \le x^2, \end{cases} \qquad (i = 1, 2, \dots, n)$$

where $\bar{y}_i(x)$ and $\overset{+}{y}_i(x)$ are single-valued and have continuous first derivatives. Suppose that E_{102} intersects, respectively at the points 1, 0, 2, the n-dimensional manifolds M^1, M^0, M^2, and that the arc E_{10} is not tangent to M^1 and M^0 at the points 1 and 0, while the arc E_{02} is not tangent to M^0 and M^2 at the points 0 and 2. Assume that M^1, M^0, M^2 are given by the equations

$$\begin{aligned}
M^1: \quad & x = x^1(\alpha_1, \alpha_2, \dots, \alpha_n) \equiv x^1(\alpha), \quad y_i = y_i^1(\alpha_1, \alpha_2, \dots, \alpha_n) \equiv y_i^1(\alpha), \\
M^0: \quad & x = x^0(\beta_1, \beta_2, \dots, \beta_n) \equiv x^0(\beta), \quad y_i = y_i^0(\beta_1, \beta_2, \dots, \beta_n) \equiv y_i^0(\beta), \\
M^2: \quad & x = x^2(\gamma_1, \gamma_2, \dots, \gamma_n) \equiv x^2(\gamma), \quad y_i = y_i^2(\gamma_1, \gamma_2, \dots, \gamma_n) \equiv y_i^2(\gamma),
\end{aligned}$$

where α, β, γ range over bounded and closed sets $T_\alpha, T_\beta, T_\gamma$, and the functions $x^1(\alpha), y_i^1(\alpha), x^0(\beta), y_i^0(\beta), x^2(\gamma), y_i^2(\gamma)$ belong to the class $C^{(3)}$ for $\alpha \in T_\alpha,\ \beta \in T_\beta,\ \gamma \in T_\gamma$, and the points $1, 0, 2$ of the curve E_{102} correspond to the values $\alpha_h = 0,\ \beta_h = 0,\ \gamma_h = 0\ (h = 1, 2, \dots, n)$. Suppose M^1, M^0, M^2 do not intersect themselves or one another, and are also regular for $\alpha \in T_\alpha,\ \beta \in T_\beta,\ \gamma \in T_\gamma$; M^1 is situated to the left, and M^2 to the right, of M^0. By $R^- + S^-$ and $R^+ + S^+$ denote the left and right subdomains into which M^0 divides $R + S$, where S^- and S^+ have a common part along M^0.

Consider the function

$$F(x, y, p) \equiv F(x, y_1, y_2, \dots, y_n, p_1, p_2, \dots, p_n) \begin{cases} F^1(x, y_1, y_2, \dots, y_n, p_1, p_2, \dots, p_n), \\ F^2(x, y_1, y_2, \dots, y_n, p_1, p_2, \dots, p_n), \end{cases}$$

$$F^1 \in C^{(4)} \quad \text{for } (x, y) \in R^- + S^-, \quad -\infty < p < +\infty;$$

$$F^2 \in C^{(4)} \quad \text{for } (x, y) \in R^+ + S^+, \quad -\infty < p < +\infty.$$

In other words, the function $F(x, y, p)$ belongs to the class $C^{(4)}$ for all $(x, y) \in R + S,\ -\infty < p < +\infty$, except for the points of the manifold M^0, at which it undergoes a discontinuity of the first kind.

Let G denote the set of functions $y_i(x)$ satisfying the following conditions: a) $y_i(x)$ are continuous; b) the curves C_{102} represented by these functions lie inside R and consist of a finite number of regular arcs; c) each curve C_{102} intersects the manifolds M^1, M^0, M^2 once, and on M^0 it has a corner point.

Formulation of the problem. Given a certain curve $E_{102} \in G$ with corner point 0. It is required to find conditions that E_{102} must satisfy in order that the functional

$$J(y) = \int_{x^1(\alpha)}^{x^0(\beta)} F^1(x, y, y')\,dx + \int_{x^0(\beta)}^{x^2(\gamma)} F^2(x, y, y')\,dx, \tag{1}$$

computed along E_{102}, have a relative minimum in the class of admissible functions G.

2. Necessary condition I (Euler condition). In order that $E_{102} \in G$ realize a minimum of the functional J, it is necessary that the arcs E_{10} and E_{02} satisfy the Euler equations in integral form.

On the arcs E_{10} and E_{02} of the curve $E_{102} \in G$ satisfying condition I, the Euler equations in differential form are valid, and at corner points (if they exist) the

Weierstrass-Erdmann conditions must be fulfilled. If along the arcs E_{10} and E_{02} one has

$$\left|F^k_{y_i' y_j'}\right| \neq 0 \quad (i, j = 1, 2, \dots, n) \tag{2}$$

($k = 1$ corresponds to the arc E_{10}, and $k = 2$ to the arc E_{02}), then on E_{10} and E_{02} the Euler equations in expanded form are also valid.

An arc E_{10} (or E_{02}) of class $C^{(2)}$, along which the expanded Euler equation is valid, is called an **extremal arc**.

One can construct a one-parameter family of admissible curves

$$y_i = y_i(x, a) \begin{cases} \overline{y}_i(x, a), & x^1(a) \leq x \leq x^0(a), \\ \overset{+}{y}_i(x, a), & x^0(a) \leq x \leq x^2(a), \end{cases} \quad (i = 1, 2, \dots, n) \tag{3}$$

containing the curve E_{102} for $a = 0$, where the functions $\overline{y}_i(x, a)$ and $\overset{+}{y}_i(x, a)$ satisfy the primary conditions for the endpoints

$$x = x^1(a) \equiv x^1[\alpha(a)], \qquad \overline{y}_i[x^1(a), a] \equiv y_i^1[\alpha(a)],$$

$$x = x^0(a) \equiv x^0[\beta(a)], \qquad \overline{y}_i[x^0(a), a] \equiv y_i^0[\beta(a)] \equiv \overset{+}{y}_i[x^0(a), a], \tag{4}$$

$$x = x^2(a) \equiv x^2[\gamma(a)], \qquad \overline{y}_i[x^2(a), a] \equiv y_i^2[\gamma(a)].$$

Along the curves (3) the functional J is transformed into a function $J(a)$,

$$J'(0) = 0, \tag{5}$$

if E_{102} realizes a minimum of the functional J. From equality (5) we obtain identities with respect to the differentials $d\alpha_k$, $d\beta_k$, $d\gamma_k$ $(k = 1, 2, \dots, n)$:

$$\left(F^1 - \overline{y}_{ix} F^1_{y_i'}\right) dx^1 + F^1_{y_i'}\, dy_i^1 = 0, \tag{6}$$

$$\left(F^2 - \overset{+}{y}_{ix} F^2_{y_i'}\right) dx^2 + F^2_{y_i'}\, dy_i^2 = 0, \tag{7}$$

$$\left[\left(F^1 - \overset{-}{y}_{ix} F^1_{y_i'}\right) - \left(F^2 - \overset{+}{y}_{ix} F^2_{y_i'}\right)\right] dx^0 + \left(F^1_{y_i'} - F^2_{y_i'}\right) dy_i^0 = 0. \tag{8}$$

Necessary condition I′. In order that the curve $E_{102} \in G$ realize a minimum of the functional J, it is necessary that at the points $1, 2$, and 0 there be satisfied, respectively, the primary transversality conditions (6), (7), and the primary corner condition (8).

Necessary condition II (the Weierstrass condition):

$$\mathcal{E}^k(x, y, y', Y') \geq 0$$

($k = 1$ corresponds to the arc E_{10}, and $k = 2$ to the arc E_{02}).

Necessary condition III (the Legendre condition):

$$F^k_{y'_i y'_j}(x, y, y')\, \pi_i \pi_j \geq 0 \qquad (i, j = 1, 2, \ldots, n)$$

($k = 1$ corresponds to the arc E_{10}, and $k = 2$ to the arc E_{02}).

3. A curve $E_{102} \in G$ with arcs E_{10} and E_{02} of class $C^{(2)}$, along which the Euler equations are satisfied in developed form, and at the point 0 the primary corner condition is satisfied, is called a **broken extremal** of the functional J. If, in addition, condition (2) is satisfied, then E_{102} is called a **proper broken extremal**.

A curve E_{102} realizing a minimum of the functional J is a broken extremal.

Theorem 1. If: 1) at the point 1 of the manifold M^1, where it intersects the proper broken extremal E_{102}, condition (6) is satisfied; 2) E_{102} at the points 1 and 0 is not tangent to M^1 and M^0 (the nontangency condition), then there exists an n-parameter family of broken extremals

$$y_i = y_i(x, \alpha_1, \alpha_2, \ldots, \alpha_n) \equiv$$

$$\equiv y_i(x, \alpha) \begin{cases} \overset{-}{y}_i(x, \alpha), & x^1(\alpha) \leq x \leq x^0(\beta), \\ \overset{+}{y}_i(x, \alpha), & x^0(\beta) \leq x \leq x^2(\gamma), \end{cases} \qquad (i = 1, 2, \ldots, n) \qquad (9)$$

intersecting transversally the manifold M^1 in a neighborhood of the point 1, and possessing the properties: a) the family (9) contains E_{102} for $\alpha = 0$; b) the functions $\overset{-}{y}_i(x, \alpha)$, $\overset{+}{y}_i(x, \alpha)$ and their first and second derivatives with respect to x have continuous partial derivatives up to the second order at the points (x, α) lying in some neighborhood of the corresponding points for E_{10} and E_{02}; c) the determinant

$$\Delta(x, \alpha) \begin{cases} \overset{-}{\Delta}(x, \alpha) = \left|\overset{-}{y}_{i\alpha_k}(x, \alpha)\right| & \text{for the arc } E_{10}, \\ \overset{+}{\Delta}(x, \alpha) = \left|\overset{+}{y}_{i\alpha_k}(x, \alpha)\right| & \text{for the arc } E_{02} \end{cases} \qquad (k, i = 1, 2, \ldots, n) \quad (10)$$

is not identically equal to zero along E_{102}.

For the proof one must first construct an n-parameter family of extremals transversal to the manifold M^1 ((4), p. 90). Next, using the corner condition at the point 0, we can construct an additional family which, together with the first, forms the desired family of broken extremals.

Definition of a focal point. Points on the broken extremal E_{102} which correspond to the zeros of the determinant $\Delta(x, \alpha)$ will be called **focal points of the manifold** M^1 on E_{102}, and the multiplicity of this zero—the **multiplicity of the focal point**. Focal points for M^2 are defined analogously.

4. The necessary Jacobi condition in terms of focal points is proved with the aid of the theorem on the envelope. For this purpose we introduce into (9)

some functions $\alpha_k(t)$, for which $\alpha_k(0) = 0$. Then we obtain a one-parameter family of broken extremals

$$y_i = y_i[x, \alpha(t)] = y(x,t) \begin{cases} \bar{y}_i(x,t), & x^1(t) \le x \le x^0(t), \\ \overset{+}{y}_i(x,t), & x^0(s) \le x \le x^2(t), \end{cases} \tag{11}$$

containing E_{102} for $t = 0$ and intersecting the manifold M^1 transversally. The family (11) may have an envelope D, tangent to each curve of the family (11) at $x = x(t)$. Then the equations of the envelope are written as follows:

$$x = x(t), \qquad y_i[x(t), t] = Y_i(t). \tag{12}$$

Generalizing the corresponding theorem for problems with smooth extremals (([4], p. 35), one can prove the following theorem:

Theorem 2. *Let point 3 be a focal point of the manifold* M^1 *on the nonsingular broken extremal* E_{102}*, and suppose that at point 3 the derivative of the determinant* $\Delta(x, \alpha)$ *with respect to* x *is nonzero. Then there exists a one-parameter family of broken extremals (11), transversal to the manifold* M^1*, containing* E_{102} *for* $t = 0$*, and having an envelope* D*, which is tangent to* E_{102} *at point 3. The functions* y_i, y_{ix}*, and* $x(t)$ *have continuous derivatives in a neighborhood of the values* x, t *belonging to the arcs* E_{10} *and* E_{02}.

The functions $\alpha_k(t)$, $\beta_k(t)$, $\gamma_k(t)$ $(k = 1, 2, \dots, n)$ determine, on the manifolds M^1, M^0, M^2, the curves L^1, L^0, L^2, passing, respectively, through the points 1, 0, 2; moreover $t = 0$ corresponds to these points. Take on L^1 the points 3 and 5, on L^0 the points 0_1 and 0_2, and on L^2 the points 4 and 6, to which correspond the values $t_1 < t_2$. If E_{30_14} and E_{50_26} are two broken extremals joining the points of the curves L^1, L^0, L^2 indicated in the indices, then it can be proved that the values of the integral J along these broken extremals are connected by the relation

$$J(E_{50_2 6}) - J(E_{30_1 4}) = J_2^*(L_{46}^2) - J_1^*(L_{35}^1), \tag{13}$$

where

$$J_k^* = \int \{F^k dx + (dy_i - y_i' dx) F_{y_i'}^k\}$$

($k = 1$ corresponds to the region R^-, and $k = 2$ to the region R^+), are Hilbert invariant integrals for the discontinuous variational problem.

Envelope theorem for a discontinuous problem. *If the envelope D of the one-parameter family of extremals (11) has a branch projecting backward from its point of intersection 6 with the extremal E_{102}, then for any point 4 on D, preceding point 6 and lying near it, the curve $E_{30_1 4} + D_{46} + E_{62}$ is admissible and*

$$J(E_{30_1 4} + D_{46} + E_{62}) = J(E_{102}). \tag{14}$$

Necessary condition IV (Jacobi condition). *In order that the nonsingular extremal E_{102}, having the single corner point 0, realize a minimum of the functional J, it is necessary that between points 1 and 2 on E_{102} there be no focal points of the manifolds M^1 and M^2.*

This assertion is proved under certain known restrictions. The Jacobi condition can be proved without these restrictions (for the planar case see (5)).

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