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1961

SovietRxiv

Abstract

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HYDROMECHANICS

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ON AN EXACT SOLUTION OF THE EQUATIONS OF ONE-DIMENSIONAL RELATIVISTIC HYDRODYNAMICS WITH A JUMP IN THE TRANSFORMATION OF THE REST MASS OF MATTER

(Presented by Academician L. I. Sedov on 1 XI 1960)

In paper ([1]) we established algebraic integrals of the relativistic equations of motion of an ideal continuous medium

$$\frac{\partial}{\partial x^k}\left[(\varepsilon+p)u^i u^k - \delta^{ik}p\right] = 0,$$

$$\frac{\partial}{\partial x^k}(\rho u^k) = 0 \qquad (i, k = 0, 1, 2, 3),$$

where ε is the internal rest energy of a unit volume of the medium; p is the pressure; ρ is the rest density; u^i is the 4-velocity of a particle.

Let us consider the case of plane one-dimensional motion, when among the constants defining the problem, in addition to the speed of light in vacuum c, there is one more constant a with dimension $[a] = ML^{-2}$. We put, in accordance with the general theory of dimensions ([2]),

$$\rho = ar^{-1}R(\lambda), \qquad p = ac^2 r^{-1}P(\lambda),$$

$$\frac{dr}{dt} = v = cV(\lambda), \qquad \lambda = \frac{r}{ct}, \tag{1}$$

where R, P, V are dimensionless functions; r is the distance from the origin of coordinates; t is time. In the case under consideration there exist three integrals that give the general solution of the plane self-similar problem. If the equation of state of the medium has the form

$$\varepsilon = \rho c^2 + \frac{p}{\gamma - 1}, \qquad \gamma = \frac{c_p}{c_v},$$

then the above-mentioned integrals can be written in the form:

mass integral *

$$R(V-\lambda)/\lambda\sqrt{1-V^2}=a_1; \tag{2}$$

energy integral

$$P\left\{\left(\frac{R}{P}+\frac{\gamma}{\gamma-1}\right)(\lambda-V)-\lambda(1-V^2)\right\}/\lambda(1-V^2)=a_2; \tag{3}$$

momentum integral

$$P\left\{\left(\frac{R}{P}+\frac{\gamma}{\gamma-1}\right)V(\lambda-V)-1+V^2\right\}/\lambda(1-V^2)=a_3, \tag{4}$$

- The mass integral in ([1]), formula (2, 3), is written with an error. The correct form is

$$\lambda^{\nu-k-3}\left[(\nu-k-3-s)M+\frac{\sigma_v R(V-\lambda)}{\lambda^{s+1}\sqrt{1-\beta^2V^2}}\right]=\text{const}.$$

where a_1, a_2, a_3 are dimensionless constants. From relations (2)–(4) we obtain

$$R=a_1\lambda\sqrt{1-V^2}/(V-\lambda); \tag{5}$$

$$P=a_1(\gamma-1)\frac{\lambda\sqrt{1-V^2}\left(1+b_2\sqrt{1-V^2}\right)}{\lambda(\gamma-1)V^2-\gamma V+\lambda}; \tag{6}$$

$$\lambda=\frac{(1-\gamma)\sqrt{1-V^2}+b_2(1-\gamma-V^2)+b_3\gamma V}{(1-\gamma)V\sqrt{1-V^2}+b_3[(\gamma-1)V^2+1]-b_2\gamma V}, \tag{7}$$

where $b_2=a_2/a_1,\ b_3=a_3/a_1$. If the initial conditions are taken as

$$\lambda=\infty\quad(t=0),\qquad V=0,\qquad R=1,\qquad P=1, \tag{8}$$

then $b_3=0,\ a_1=-1,\ b_2=-\gamma/(\gamma-1),\ a_2=\gamma/(\gamma-1),\ a_3=0$, and the solution in the neighborhood of $\lambda=\infty$ (region 1) has the form

$$R=\lambda\sqrt{1-V^2}/(\lambda-V); \tag{9}$$

$$P = (\gamma - 1)\frac{\lambda\sqrt{1 - V^2}\left(\frac{\gamma}{\gamma - 1}\sqrt{1 - V^2} - 1\right)}{\lambda(\gamma - 1)V^2 - \gamma V + \lambda}; \tag{10}$$

$$\lambda = \frac{\gamma(\gamma - 1) + \gamma V^2 - (\gamma - 1)^2\sqrt{1 - V^2}}{\gamma^2 V - (\gamma - 1)^2 V\sqrt{1 - V^2}}. \tag{11}$$

This solution does not continue continuously to the center of symmetry. To extract from the general solution the curve $V = V(\lambda)$ passing through the point $\lambda = 0,\ V = 0$, it is necessary in (7) to put $b_2 = -1$. The other particular solution obtained as a result must be joined to the solution (9)–(11), satisfying the conditions at the discontinuity. It is easy to see, however, that the constants a_1, a_2, a_3 of the integrals (2)–(4) coincide exactly with the values of the fluxes of rest mass, energy, and momentum through the surface of discontinuity.

Instead of the condition of conservation of rest mass, consider the relation

$$\frac{\rho_1(v_1 - D)}{\sqrt{1 - v_1^2/c^2}} = \frac{\rho_2(v_2 - D)}{\sqrt{1 - v_2^2/c^2}} + q, \qquad v = cV(\lambda), \tag{12}$$

where D is the velocity of the surface of discontinuity, and q is a physical characteristic of the process. To preserve self-similarity we must put

$$q = \rho_1 c f\left(\frac{p_1}{\rho_1 c^2}\right)/\sqrt{1 - v_1^2/c^2},$$

where f is a dimensionless function. In dimensionless variables (12) is written as

$$a_1\lambda^* = \frac{R_1(V_1 - \lambda^*)}{\sqrt{1 - V_1^2}} = \frac{R_2(V_2 - \lambda^*)}{\sqrt{1 - V_2^2}} + Q(\lambda^*)\lambda^*, \tag{13}$$

where λ^* is the value of λ at the jump, and

$$\lambda^* Q(\lambda^*) = \frac{R_1 f(P_1/R_1)}{\sqrt{1 - V_1^2}}. \tag{14}$$

Then, if on both sides of the jump the motion is adiabatic, then

$$R_2(V_2 - \lambda^*) = \alpha_1\lambda^*\sqrt{1 - V_2^2},$$

where α_1 is an integration constant. Replacing in (5)–(7) the constants a_1, b_2, b_3, respectively, by $\alpha_1,\ \beta_2 = a_2/\alpha_1,\ \beta_3 = a_3/\alpha_1$, we obtain, on the basis of (13) and the fact that for the curve $V = V(\lambda)$ passing through the point $\lambda = 0,\ V = 0,\ \beta_2 = -1$,

$$\alpha_1 = -a_2 = a_1 - Q(\lambda^*),$$

whence

$$Q(\lambda^*) = a_1 + a_2 = \frac{1}{\gamma - 1}, \qquad \alpha_1 = -\frac{\gamma}{\gamma - 1}, \qquad \beta_3 = 0. \tag{15}$$

The properties of the gas after the discontinuity has passed may change. Therefore in (7) γ should, generally speaking, be replaced by $\gamma_1 \neq \gamma$. Then, on the other side of the jump (region 2), we shall have

$$R = \frac{\gamma\lambda\sqrt{1-V^2}}{(\gamma-1)(\lambda - V)}; \tag{16}$$

$$P = \frac{\gamma(\gamma_1 - 1)\lambda\sqrt{1-V^2}\,(\sqrt{1-V^2} - 1)}{(\gamma-1)[\lambda(\gamma_1-1)V^2 - \gamma_1 V + \lambda]}; \tag{17}$$

$$\lambda = \frac{(\gamma_1 - 1)(1 - \sqrt{1-V^2}) + V^2}{V[\gamma_1 - (\gamma_1 - 1)\sqrt{1-V^2}]}. \tag{18}$$

Relation (14), with (15) taken into account, serves to determine the velocity of motion of the discontinuity surface λ^*:

$$\lambda^* = V_1 + (\gamma - 1) f\left(\frac{P_1}{R_1}\right).$$

Figures 1 and 2 present graphs of the functions V, P, R as functions of λ for the values $\gamma = 7/5,\ \gamma_1 = 5/3$, calculated from formulas (9)–(11)

Fig. 1: Graphs of functions V, P, R as functions of lambda.

Fig. 1

Fig. 2: Graphs of functions V, P, R as functions of lambda.

Fig. 2

and (16)–(18). If $f(P_1/R_1) = 5/4,\ \lambda^* = 0.9$. The solution obtained corresponds, in nonrelativistic hydrodynamics, to the following exact solution of the Euler equations:

in region 1,

$$v = \frac{b}{\gamma+1}\left[\lambda - \sqrt{\lambda^2 - 2(\gamma+1)}\right], \quad \rho = \frac{a\lambda(\gamma+1)}{r[\gamma\lambda + \sqrt{\lambda^2 - 2(\gamma+1)}]}, \quad p = \frac{av}{t}; \tag{19}$$

in region 2,

$$v = \frac{2}{\gamma_1 + 1} \frac{r}{t}, \quad \rho = \frac{a(\gamma_1 + 1)}{(\gamma_1 - 1)r}, \quad p = \frac{2ar}{(\gamma_1 + 1)t^2}, \tag{20}$$

where a and b are constants with dimensions $[a] = ML^{-2}, [b] = LT^{-1}$, $\lambda = r/bt$. Solutions (19) and (20) arise from (9)–(11) and (16)–(18), respectively, in the limit as $b/c \to 0$. At the jump, the conservation conditions for the fluxes of mass and momentum are then satisfied. The energy-conservation condition shows that, when a particle passes through the jump from region 1 into region 2, the sum of the kinetic energy of relative motion and the heat content of the particle decreases by the amount $b^2/(\gamma - 1)$ per unit mass of matter.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
1 XI 1960

CITED LITERATURE

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2. L. I. Sedov, *Methods of Similarity and Dimensionality in Mechanics*, Moscow, 1957.

Note: Figure translations are in progress. See original paper for figures.