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Abstract

Full Text

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ON A LIMIT THEOREM

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The purpose of the present note is to prove the following limit theorem.

Theorem 1. Let $\xi_1, \xi_2, \ldots, \xi_k, \ldots$ be a sequence of mutually independent random variables, given, generally speaking, by nonpositive probability densities $p_k(x)$ $(k = 1, 2, \ldots)$, satisfying the conditions:

1) For every k $(k = 1, 2, \ldots)$

$$\int p_k(x)\,dx = 1, \quad \int x p_k(x)\,dx = 0, \ldots, \quad \int x^{2q-1} p_k(x)\,dx = 0,$$

$$\int x^{2q} p_k(x)\,dx = (-1)^{q+1} b_k^{2q}.$$

2) There exists a constant C such that for every n the inequality

$$\sum_{k=1}^{n} \int x^{2q} |p_k(x)|\,dx < C \sum_{k=1}^{n} \left| \int x^{2q} p_k(x)\,dx \right|.$$

is valid.

3) For any $\lambda > 0$ the limit

$$\lim_{n\to\infty} \frac{1}{B_n^{2q}} \sum_{k=1}^{n} \int_{|x|>\lambda B_n} x^{2q} |p_k(x)|\,dx = 0,$$

where

$$B_n^{2q} = \sum_{k=1}^{n} b_n^{2q}.$$

4) The limit

$$\lim_{\varepsilon\to 0}\max_{\substack{1\le k\le n\\ 1\le n<\infty}}\frac{1}{B_n^{2q}}\int_{|x|\le\varepsilon B_n} x^{2q}|p_k(x)|\,dx=0.$$

Then, if $P_n(x)$ is the probability density of the normalized sum

$$\zeta_n=\frac{1}{B_n}\sum_{k=1}^{n}\xi_k,$$

then there exists the limit $P(x)$

$$P(x)=\lim_{n\to\infty}P_n(x)=\frac{1}{2\pi}\int e^{-t^{2q}/(2q)!+itx}\,dt.$$

The limit is understood in the sense of weak convergence on the space Z.

Before proceeding to the proof, let us note that while condition 1) pertains rather to the formulation of the problem, conditions 2) and 3) are analogues of the usual conditions of boundedness of the variance of the sum and of the Lindeberg condition (see, for example, ([1],[2])), condition 4) has no analogue and is connected with the nonpositivity of the probability densities $p_k(x)$.

We now indicate the proof of Theorem 1. For convenience we introduce new notation. Let $p_{nk}(x)$ be the probability density of the random variable

$$\xi_{nk}=\frac{1}{B_n}\xi_k\quad(1\le k\le n;\ n=1,2,...).$$

Conditions 1)–4) of Theorem 1 in this notation take the form:

1) For all k $(1\le k\le n)$ and $n=1,2,...,$

$$\int p_{nk}(x)\,dx=1,\qquad \int xp_{nk}(x)\,dx=0,\dots,\qquad \int x^{2q-1}p_{nk}(x)\,dx=0,$$

$$\int x^{2q}p_{nk}(x)\,dx=(-1)^{q+1}\frac{b_k^{2q}}{B_n^{2q}}.$$

2) There exists a constant C such that, for every n,

$$\sum_{k=1}^{n}\int x^{2q}\,|p_{nk}(x)|\,dx<C.$$

3) For every $\lambda>0$ the limit

$$\lim_{n\to\infty}\sum_{k=1}^{n}\int_{|x|>\lambda}x^{2q}\,|p_{nk}(x)|\,dx=0.$$

4) The limit

$$\lim_{\varepsilon\to 0}\max_{\substack{1\le k\le n\\ 1\le n<\infty}}\int_{|x|\le\varepsilon} x^{2q}\,|p_{nk}(x)|\;dx=0.$$

Denote by $f_{nk}(t)$ the characteristic function of ξ_{nk}, and by

$$\varphi_n(t)=\prod_{k=1}^{n} f_{nk}(t)$$

the characteristic function of the sum

$$\zeta_n=\frac{1}{B_n}\sum_{k=1}^{n}\xi_k=\sum_{k=1}^{n}\xi_{nk}.$$

We shall show that for $|t|\le T$, uniformly in k $(1\le k\le n)$, the limit

$$\lim_{n\to\infty} f_{nk}(t)=1.$$

Indeed, by condition 1),

$$f_{nk}(t)-1=\int\left(e^{itx}-1-itx-\cdots-\frac{(itx)^{2q-1}}{(2q-1)!}\right)p_{nk}(x)\,dx,$$

therefore

$$|f_{nk}(t)-1|\le\frac{t^{2q}}{(2q)!}\int x^{2q}\,|p_{nk}(x)|\;dx.$$

Splitting the domain of integration into $|x|\le\varepsilon$ and $|x|>\varepsilon$, we note that the integral over $|x|\le\varepsilon$ can be made, by condition 4), arbitrarily small by choosing $\varepsilon>0$ (uniformly in n), while the integrals over $|x|>\varepsilon$ tend to zero as $n\to\infty$ by condition 3).

Consequently, $f_{nk}(t)\to 1$ for $|t|\le T$ and as $n\to\infty$, uniformly in k $(1\le k\le n)$. In particular, for $|t|\le T$ and sufficiently large n the inequality

$$\max_{1\le k\le n}|f_{nk}(t)-1|<\frac{1}{2}$$

will hold.

Let

$$\ln\varphi_n(t)=\sum_{k=1}^{n}(f_{nk}(t)-1)+R_n,$$

where

$$R_n=\sum_{k=1}^{n}\sum_{s=2}^{\infty}\frac{(-1)^s}{s}\left(f_{nk}(t)-1\right)^s.$$

For sufficiently large n, by virtue of the last inequality, the estimate is valid···

$$|R_n| \leq \sum_{k=1}^{n} \sum_{s=2}^{\infty} \frac{1}{2} |f_{nk}(t) - 1|^s = \frac{1}{2} \sum_{k=1}^{n} \frac{|f_{nk}(t) - 1|^2}{1 - |f_{nk}(t) - 1|} <$$

$$< \sum_{k=1}^{n} |f_{nk}(t) - 1|^2 \leq \max_{1 \leq k \leq n} |f_{nk}(t) - 1| \sum_{k=1}^{n} |f_{nk}(t) - 1|.$$

But since, by condition 2),

$$\sum_{k=1}^{n} |f_{nk}(t) - 1| \leq \frac{t^{2q}}{(2q)!} \sum_{k=1}^{n} \int x^{2q} |p_{nk}(x)|\, dx < C \frac{t^{2q}}{(2q)!},$$

we have $R_n \to 0$ $(n \to \infty)$ uniformly with respect to k $(1 \leq k \leq n)$ for $|t| \leq T$. Let now

$$\sum_{k=1}^{n} (f_{nk}(t) - 1) = -\frac{t^{2q}}{(2q)!} + \rho_n,$$

where

$$\rho_n = \frac{t^{2q}}{(2q)!} + \sum_{k=1}^{n} \int \left(e^{itx} - 1 - itx - \cdots - \frac{(itx)^{2q-1}}{(2q-1)!} \right) p_{nk}(x)\, dx.$$

By condition 1),

$$\rho_n = \sum_{k=1}^{n} \int \left(e^{itx} - 1 - itx - \cdots - \frac{(itx)^{2q}}{(2q)!} \right) p_{nk}(x)\, dx = \Sigma_1 + \Sigma_2,$$

where in Σ_1 the integration is carried out over the region $|x| \leq \varepsilon$, and in Σ_2, over $|x| > \varepsilon$. Since $|\rho_n| \leq |\Sigma_1| + |\Sigma_2|$, we estimate Σ_1 and Σ_2 in modulus. The following estimates are valid:

$$|\Sigma_1| \leq \frac{t^{2q+1}}{(2q+1)!} \varepsilon \sum_{k=1}^{n} \int_{|x| \leq \varepsilon} x^{2q} |p_{nk}(x)|\, dx \leq \frac{t^{2q+1}}{(2q+1)!} \varepsilon C \quad \text{by condition 2);}$$

$$|\Sigma_2| \leq 2 \frac{t^{2q}}{(2q)!} \sum_{k=1}^{n} \int_{|x| > \varepsilon} x^{2q} |p_{nk}(x)|\, dx \to 0 \quad (n \to \infty) \quad \text{by condition 3).}$$

Consequently, for any $|t| \leq T$, $\lim_{n \to \infty} \rho_n = 0$, and hence

$$\lim_{n\to\infty} \ln \varphi_n(t) = -\frac{t^{2q}}{(2q)!}.$$

And since

$$P_n(x) = \frac{1}{2\pi}\int e^{itx}\varphi_n(t)\,dt,$$

it follows from what has been proved that $P_n(x) \to P(x)$ $(n \to \infty)$ in the sense of weak convergence over Z (see (^{3},^{4})).

To obtain stronger convergence of the functions $P_n(x)$, it is necessary, in addition to conditions 1)–4) of Theorem 1, to impose conditions analogous to sufficient conditions for the stability of difference schemes (see, for example, (^{5})). Thus, for example, the following is true.

Theorem 2. Let, for $p_k(x)$ $(k = 1, 2, \ldots)$, conditions 1)–4) of Theorem 1 be satisfied and suppose, in addition, that the inequality

$$5) \qquad |f_{nk}(t)| \le \left(1 + \frac{A}{n}\right)\left(\frac{n}{n+|t|}\right)^{\alpha} \qquad (k = 1, \ldots, n)$$

holds for all k, t $(-\infty < t < \infty)$ and for some $\alpha > 0$. Then $P_n(x) \to P(x)$ $(n \to \infty)$ in the mean.

Indeed, in this case there exists, for all t, the limit $\lim\limits_{n\to\infty} \varphi_n(t)$, since

$$\varphi_n(t) = \prod_{k=1}^{n} f_{nk}(t),$$

and moreover this limit belongs to L_2. In addition, since the assertion of Theorem 1 remains in force, we have

$$\lim_{n\to\infty} \varphi_n(t) = \exp\left\{-\frac{t^{2q}}{(2q)!}\right\}.$$

From this, by virtue of the isometry in L_2 of the Fourier transform operator, the assertion of Theorem 2 follows.

Let us note some obvious properties of the limiting probability density $P(x)$.

Property 1. Just as the fundamental solution of the heat-conduction equation for any t is the Gaussian law, the solution of the equation

$$\frac{\partial u}{\partial t} = (-1)^{q+1}\frac{\partial^{2q} u}{\partial x^{2q}}$$

for any t has the form of the limiting probability density $P(x)$ considered above.

Property 2. The limiting law determined by the probability density $P(x)$ is infinitely divisible.

Property 3. The limiting law determined by the probability density $P(x)$ is stable.

The theorems formulated establish a connection between the central limit theorem of probability theory and the limiting behavior of difference schemes approximating differential equations (see also ([6])).

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Note: Figure translations are in progress. See original paper for figures.