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Abstract

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A Phragmén-Lindelöf Type Theorem for a Linear Elliptic System

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The well-known theorem stating that a harmonic function $u(x, y)$, bounded on the sides of the strip $x > 0,\ 0 < y < 1$ and satisfying, for $0 < y < 1$, the condition $u(x, y) = O(e^{(\pi-\varepsilon)x})$, $x \to \infty$, is bounded in the entire strip, has recently received very broad generalizations. The most significant works are those of E. M. Landis ([1]), P. D. Lax ([2]), and M. A. Evgrafov ([3]). However, the results mentioned, although they bear the common name "Phragmén-Lindelöf type theorem," are not exact analogues of the classical principle, which, as applied to the Cauchy-Riemann system

$$\frac{\partial}{\partial y}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u \\ v \end{pmatrix}$$

is formulated as follows: if a harmonic function $u(x, y)$ and its conjugate $v(x, y)$ are bounded on the sides of the strip $x > 0,\ 0 < y < 1$, and inside it satisfy the condition $|u + iv| = O(\exp\exp(\rho - \varepsilon)x)$, $x \to \infty$, then $|u + iv| = O(1)$, $x \to \infty$.

One may say that results of the first type concern the solution of the Dirichlet problem, while those of the second type concern the solution of the Cauchy problem. There are comparatively few of the latter in the literature. Namely, these are the papers of M. A. Evgrafov and I. A. Chegis ([4]) and I. A. Chegis ([5]), where the question of harmonic functions of three variables in circular ([4]) and rectangular ([5]) half-cylinders is solved. In addition, one general theorem for the case of the whole plane was proved by G. E. Shilov ([6]).

It is therefore of known interest to obtain, by a method different from those used in ([4,5]) and ([6]), a generalization of the Phragmén-Lindelöf principle for a half-strip to solutions of systems of the form

$$\frac{\partial \hat{u}}{\partial y} = A\frac{\partial \hat{u}}{\partial x} + P\hat{u}.$$

Here A, P are constant matrices of order n; $\hat{u}(x, y)$ is an n-dimensional vector.

Theorem. Let $\hat{u}(x, y)$ be a solution of the equation

$$\frac{\partial \hat{u}}{\partial y} = A\frac{\partial \hat{u}}{\partial x} + P\hat{u} \tag{1}$$

in the half-strip $x > 0,\ 0 < y < 1$, continuous together with its partial derivatives up to the boundary. If: 1) equation (1) is elliptic, i.e. all eigenvalues of the matrix A have nonzero imaginary part; 2) there exists a function $\varphi(x) > 0$ satisfying the condition

$$\lim_{x\to\infty} \frac{\varphi'(x)}{\varphi(x)} = 0 \tag{2}$$

and such that for $y = 0$ and $y = 1$

$$\|\hat{u}(x,y) - \hat{u}(s,y)\| \le |x-s|^\alpha \{\varphi(x) + \varphi(s)\}; \quad \|\hat{u}(x,y)\| \le \varphi(x); \quad \alpha > 0; \tag{3}$$

3) for any fixed $y,\ 0 < y < 1$,

$$\hat{u}(x,y) = O(\exp\exp(\rho-\varepsilon)x), \quad x\to\infty, \quad \varepsilon > 0 \text{ arbitrary}, \tag{4}$$

where $\rho = \pi/\max|\operatorname{Im} a_k|$, a_k are the eigenvalues of the matrix A, then uniformly in all y

$$\hat{u}(x,y) = O(\varphi(x)), \quad x\to\infty. \tag{5}$$

The proof, in general outline, is as follows. Put

$$\hat{v}(x,y) = \int_{-x}^{\infty} u(x+t,y)e^{-\varepsilon\rho_1 t+\rho_1 t}\,dt = \int_0^{\infty} \hat{u}(t,y)e^{-\varepsilon\rho_1(t-x)+\rho_1(t-x)}\,dt, \tag{6}$$

$$\rho > \rho_1 > \rho - \varepsilon.$$

Then $\hat{v}(x,y)$ satisfies the equation

$$\frac{\partial \hat{v}}{\partial y} = A\frac{\partial \hat{v}}{\partial x} + P\hat{v} + h(x)\hat{f}(y); \qquad h(x) = e^{-\varepsilon-\rho_1 x-\rho_1 x}, \qquad \hat{f}(y) = -A\hat{u}(0,y). \tag{7}$$

The main task is to estimate $\hat{v}(x,y)$. To this end let us consider the functions $\hat{v}(x,0)$ and $\hat{v}(x,1)$. From (6) it is seen that in the plane $\xi = x + i\lambda$ they are continued from the real axis into the domain $|\operatorname{Im}\xi| = |\lambda| \le \pi/2\rho + \delta$, $\delta > 0$. Moreover, in this domain

$$\|\hat{v}(x+i\lambda,0)\| + \|\hat{v}(x+i\lambda,1)\| \le \int_0^\infty 2\varphi(t)\left|e^{-\varepsilon\rho_1(t-x-i\lambda)+\rho_1(t-x-i\lambda)}\right| dt \le$$

$$\le \text{const} \cdot e^{-c_1 e^{-\rho_1 x} + c_2 x}. \qquad (8)$$

Here $c_2 > 0$ is arbitrary, $c_1 = \frac{1}{2}\cos[(\pi/2\rho + \delta)\rho_1] > 0$.

If now we denote

$$\tilde{v}(z,0) = \int_{-\infty}^{\infty} \hat{v}(x,0)e^{-xz}\,dx, \qquad \tilde{v}(z,1) = \int_{-\infty}^{\infty} \hat{v}(x,1)e^{-xz}\,dx, \qquad \operatorname{Re} z > 0, \qquad (9)$$

then from the preceding estimate, with the aid of standard considerations from the theory of the Fourier transform, we easily obtain

$$\|\tilde{v}(z,0)\| + \|\tilde{v}(z,1)\| \le c_a e^{-(\pi/2\rho+\delta)|z|}, \qquad a = \operatorname{Re} z = \text{const} > 0. \qquad (10)$$

We reconstruct the function $\hat{v}(x,y)$ for $0 \le y \le \frac{1}{2}$ by means of the formula

$$\hat{w}_0(x,y) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \left\{ e^{zA(z)y}\tilde{v}(z,0) + \frac{1}{\rho_1}\Gamma\left(\frac{z}{\rho_1}+1\right)\int_0^y e^{zA(z)(y-t)}\hat{f}(t)\,dt \right\} e^{xz}\,dz, \qquad (11)$$

where, as above, $\hat{f}(t) = -A\hat{u}(0,t)$, and $zA(z) \equiv zA + P$. Let us investigate the growth of the various expressions from which the integrand in (11) is composed on the contour $\operatorname{Re} z = a$. We have

$$\|e^{zA(z)y}\| \le c_a(1+|z|^n)e^{\max \operatorname{Re}(za_k(z))y}. \qquad (12)$$

Here $a_k(z)$ denote the eigenvalues of the matrix $A(z) = A + \frac{1}{z}P$. Taking into account that, as $|z| \to \infty$, $a_k(z) = a_k(1+o(1))$, and also that $0 \le y \le \frac{1}{2}$ and $\max|\operatorname{Im} a_k| = \pi/\rho$, we find

$$\|e^{zA(z)y}\| \le c_a e^{(\pi/2\rho+\varepsilon)|z|}. \qquad (13)$$

Consequently,

$$\left\|\int_0^y e^{zA(z)(y-t)}\hat{f}(t)\,dt\right\| \le c_a e^{(\pi/2\rho+\varepsilon)|z|}. \qquad (14)$$

Finally, by Stirling' s formula,

$$\Gamma\left(\frac{z}{\rho_1}+1\right) \le c_a e^{-(\pi/2\rho_1-\varepsilon)|z|} \le c_a c^{-(\pi/2\rho+\delta)|z|}. \tag{15}$$

Estimates (10) and (13)–(15) show that on the contour $\operatorname{Re} z = a$ the integrand has the form $O(e^{-(\delta-\varepsilon)|z|})$. Hence the absolute convergence of the integral (11) follows, as well as the possibility of differentiating under its sign with respect to x and y, and the independence of $\hat{w}_0(x, y)$ of a for $a > 0$. From this also follows the inequality

$$\|\hat{w}_0(x,y)\| \le c_a e^{ax}, \qquad 0 \le y \le \tfrac{1}{2}, \qquad a > 0 \text{ arbitrary.} \tag{16}$$

It remains to prove that $\hat{w}_0(x, y)$ and $\hat{v}(x, y)$ coincide. Denote $L = \dfrac{\partial}{\partial y} - A\dfrac{\partial}{\partial x} - P$. Then, in view of (7), $L(\hat{v} - \hat{w}_0) = 0$, and, obviously, $\hat{v}(x, 0) = \hat{w}_0(x, 0)$. Combining now the known Holmgren theorem on the local uniqueness of the solution of the Cauchy problem with the possibility of analytic continuation of any solution of the elliptic equation $L\hat{u} = 0$ into a certain domain in the plane $\xi = x + i\lambda$ (for fixed y, $0 < y < \frac{1}{2} + \delta$), we obtain that for $0 < y < \frac{1}{2}$ the identity $\hat{v} = \hat{w}_0$ holds.

The case $\frac{1}{2} < y < 1$ is exhausted analogously.

Thus, $\|\hat{v}(x, y)\| \le c_a e^{ax}$ for any $a > 0$. Therefore one may introduce the function

$$\tilde{v}(z,y) = \int_{-\infty}^{\infty} \hat{v}(x,y) e^{-xz}\, dx, \tag{17}$$

regular in the domain $\operatorname{Re} z > 0$. It satisfies the equation

$$\frac{d\tilde{v}}{dy} = zA(z)\tilde{v} + \frac{1}{\rho_1}\Gamma\left(\frac{z}{\rho_1}+1\right)\hat{f}(y), \qquad \hat{f}(y) = -A\hat{u}(0,y). \tag{18}$$

Consequently,

$$\tilde{v}(z,y) = e^{zA(z)y}\tilde{v}(z,0) + \frac{1}{\rho_1}\Gamma\left(\frac{z}{\rho_1}+1\right)\int_0^y e^{zA(z)(y-t)}\hat{f}(t)\, dt \tag{19}$$

or

$$\tilde{v}(z,y) = e^{zA(z)(y-1)}\tilde{v}(z,1) + \frac{1}{\rho_1}\Gamma\left(\frac{z}{\rho_1}+1\right)\int_1^y e^{zA(z)(y-t)}\hat{f}(t)\, dt. \tag{20}$$

Denote by $E^{\pm}(z)$ the projection operators onto the collection of invariant subspaces of the matrix $A(z)$ corresponding to its eigenvalues $a_k(z)$ with the condition $\operatorname{Im} a_k(z) > 0$, respectively $\operatorname{Im} a_k(z) < 0$. These matrices are regular in the ring $r_0 < z < \infty$. Put, moreover:

$$Q(z,s) = \begin{cases} E^{+}(z)e^{zA(z)s}\operatorname{sgn} s, & s \operatorname{Im} z > 0, \\ E^{-}(z)e^{zA(z)s}\operatorname{sgn} s, & s \operatorname{Im} z < 0. \end{cases}$$

If $\theta = \frac{1}{2}\min|\arg(\pm a_k)|$, $\mu = \frac{1}{2}\min|\operatorname{Im} a_k|$, then in the domain $|\arg(z-\sigma) \pm \pi/2| \le \theta$

$$\|Q(z,s)\| \le c_\theta e^{-\mu|z||s|}. \tag{21}$$

We now multiply, for $\operatorname{Im} z \ge 0$, both sides of (19) by $E^{+}(z)$, both sides of (20) by $E^{-}(z)$, and add. Noting also the obvious relation

$$\tilde{v}(z,0) = \frac{1}{\rho_1}\Gamma\left(\frac{z}{\rho_1}+1\right)\tilde{u}(z,0),$$

where $\tilde{u}(z,0)$ is the Laplace transform of the function $\hat{u}(x,0)$, put $y = +0$ and divide both sides of the equality obtained by $\frac{1}{\rho_1}\Gamma\left(\frac{z}{\rho_1}+1\right)$.

The following boundary identity is obtained:

$$\tilde{u}(z,0) = Q(z,+0)\tilde{u}(z,0) - Q(z,-1)\tilde{u}(z,1) + \int_0^1 Q(z,y-t)\hat{f}(t)\,dt. \tag{22}$$

We use it to obtain the inversion formula for the function $\hat{u}(x,y)$

$$2\pi i\,\hat{u}(x,y) = \int_{\sigma-i\infty}^{\sigma+i\infty} Q(z,y)\tilde{u}(z,0)e^{xz}\,dz - \int_{\sigma-i\infty}^{\sigma+i\infty} Q(z,y-1)\tilde{u}(z,1)e^{xz}\,dz +$$

$$+\int_{\sigma-i\infty}^{\sigma+i\infty}\left\{\int_0^1 Q(z,y-t)\hat{f}(t)\,dt\right\}e^{xz}\,dz. \tag{23}$$

Here $\sigma > 0$ is chosen so large that in the ring $|z| > \sigma$ the matrices $E^{\pm}(z)$ are regular.

The proof is carried out according to the same scheme as the estimate of the function $\hat{v}(x,y)$. Namely, denoting the right-hand side of (23) by $2\pi i\hat{w}(x,y)$, we find that this function is continuous together with its partial derivatives up to the boundary, $L(\hat{v}-\hat{w})=0$, and, according to (23),

$$\hat{w}(x,0)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\tilde{u}(z,0)e^{xz}\,dz=\hat{u}(x,0).$$

Let us also denote

$$K(x,y)=\int_{\sigma-i\infty}^{\sigma+i\infty}Q(z,y)e^{xz}\,dz,\qquad y\neq 0. \tag{24}$$

Deformation of the contour $\operatorname{Re} z=\sigma$ in the domain

$$|\arg(z-\zeta)\pm\pi/2|\leq\theta$$

gives the estimate

$$\frac{\partial^p K(x,y)}{\partial x^p}\leq \text{const}\ \frac{e^{\sigma x}}{(|x|+|y|)^{p+1}},\qquad p=0,1,\dots. \tag{25}$$

Substituting now in (23), instead of $\tilde{u}(z,0)$ and $\tilde{u}(z,1)$, their expressions through $\hat{u}(x,0)$ and $\hat{u}(x,1)$, after simple transformations we obtain an integral formula of Cauchy-form type

$$2\pi i\hat{u}(x,y)=\int_0^\infty K(x-t,y)\hat{u}(t,0)\,dt+\int_\infty^0 K(x-t,y-1)\hat{u}(t,1)\,dt+$$

$$+A\int_1^0 K(x,y-t)\hat{u}(0,t)\,dt, \tag{26}$$

valid for $x>0,\ 0<y<1$. On the basis of it and of formulas (3), (26), we easily find

$$\|\hat{u}(x,y)\|=O(e^{\sigma_1 x}),\qquad \sigma_1>\sigma>r_0,\qquad x\to\infty. \tag{27}$$

The final result is obtained from this almost directly by reference to the theorem proved in note (7). Some additional arguments are carried out in close connection with the exposition in that note.

It should also be pointed out that, in order to justify the operations performed by the Laplace transform, one has to consider, instead of $\hat{u}(x,y)$, the function

$$\hat{v}(x,y)=\frac{1}{2}\int_0^x(t-x)^2\hat{u}(t,y)\,dt,$$

which has sufficiently good differentiability properties. For it, instead of (26), a somewhat more complicated formula is proved, which, after threefold differentiation under the integral signs, reduces to (26).

In conclusion I take the opportunity to express my gratitude to M. A. Evgrafov, to whom the formulation of the problem and the main idea of the proof are due.

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Note: Figure translations are in progress. See original paper for figures.