ON THE DISTRIBUTION OF PRIME IDEALS

1961

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Abstract

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MATHEMATICS

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(Presented by Academician A. I. Mal' cev on 23 V 1961)

Yu. V. Linnik (1) proved that the least prime number of the arithmetic progression $Dl + l$ $[(D, l) = 1,\ u = 0, 1, 2, ...]$ does not exceed D^c, where c (and below c') is a suitable absolute positive constant. K. A. Rodosskii (2) gave a simpler proof, and I (3) proved the existence of a prime number of an arithmetic progression in the interval (x, xD^ε) for any positive $\varepsilon \leq c$, for all $D \geq D_0(\varepsilon)$ and all $x \geq D^{c' \log(c/\varepsilon)}$. The purpose of the present note is to report that the following results can be proved by the same method.

Theorem 1. *Let K, $\mathfrak{f}$, $\mathfrak{H}$ denote respectively any algebraic field of degree $n \geq 1$, any ideal in the field K, and any class of ideals modulo $\mathfrak{f}$. Let $D = |\Delta| N\mathfrak{f} > 1$, where Δ is the discriminant of the field and $N\mathfrak{f}$ is the norm of the ideal $\mathfrak{f}$. There exists a positive constant c, depending only on n, such that for any $x \geq 1$ there is, in the interval (x, xD^c), a prime number $p = N\mathfrak{p}$, with $\mathfrak{p} \in \mathfrak{H}$.*

For $\mathfrak{f} = \mathfrak{o}$ (the unit ideal) it follows that any class (in the ordinary sense) of ideals contains a prime ideal with norm $\leq |\Delta|^c$.

Theorem 2. *There exist absolute constants $c > 0$, $c' > 0$, such that for any positive $\varepsilon \leq c$, for all discriminants d with $|d| = D \geq D_0(\varepsilon)$, and for all $x \geq D^{c' \log(c/\varepsilon)}$, there is, in the interval (x, xD^ε), a prime number p represented by the given primitive (and positive, if $d < 0$) binary quadratic form ψ of discriminant d.*

The proof of Theorem 1 is based on Lemmas 1-3.

Lemma 1. *Let $\zeta(s, \chi)$ denote any Hecke function (4) with character modulo $\mathfrak{f}$. There exists c_1 (depending only on n) such that in the region*

$$\sigma \geq 1 - \frac{c_1}{\log D(1 + |t|)^{3/4}}, \qquad (\sigma = \operatorname{Re} s,\ t = \operatorname{Im} s,\ D > D_0 > 1) \tag{1}$$

the function $\zeta(s, \chi)$ has no zeros with complex χ. For at most one real exceptional character $\chi = \chi'$ in (1) a simple zero β of the function $\zeta(s, \chi')$ is possible; moreover $\beta = 1 - \delta$ is real and

$$\delta > D^{-2n}. \tag{2}$$

The lemma is proved by the methods of Titchmarsh ([5]) and Page ([6]). Here an important role is played by the estimate

$$F(s) \ll \delta^{-n} D^{\frac{1}{2}(1-\sigma)}(1+|t|)^{(1+\delta-\sigma)n/2}$$

$$\left(-\delta \leq \sigma \leq 1+\delta, \qquad 0<\delta \leq \min\left(\frac{1}{2}, \frac{1}{\log D}\right)\right) \tag{3}$$

of the functions $F(s) = \zeta(s,\chi)$ (if χ is not the principal character χ_0) and $F(s) = \zeta(s,\chi_0)(s-1)/(s-2)$, which is proved by applying function-

equation $\zeta(s,\chi)$, the theorem of Fragmén—Lindelöf and the theorem of Dech ([7]). If $\chi' = \chi_0$, then (2) is a consequence of the estimate

$$\operatorname*{Res}_{s=1} \zeta(s,\chi_0) = D^{o(1)} \quad (D \to \infty),$$

which in turn follows from the estimate of the residue of the Dedekind zeta-function of the field K (([8]), (16)). For $\chi' \neq \chi_0$, (2) is proved by contour integration of the function

$$D^{a(s-1)}\Gamma(s-1)\zeta(s,\chi')\zeta(s,\chi_0)'$$

(where $a \geq 2n+1$), taking into account the fact that the coefficients a_m of the expansion

$$\zeta(s,\chi')\zeta(s,\chi_0) = \sum_{m=1}^{\infty} a_m m^{-s} \quad (\sigma > 1)$$

are nonnegative and $a_m \geq 1$ for

$$m = 1^{2n},\ 2^{2n},\ 3^{2n}, \dots .$$

Lemma 2. For a suitable $A > 0$ (depending only on n) and

$$\lambda_0 = A \log \frac{eA}{\delta_0 \log D}, \qquad \delta_0 = \begin{cases} \delta, & \text{if } \delta \leq A/\log D, \\ A/\log D & \text{otherwise,} \end{cases}$$

the rectangle

$$(1 - \lambda_0/\log D \leq \sigma \leq 1,\ |t| \leq D)$$

contains no zeros $\rho \neq 1-\delta$ of the function $\prod_\chi \zeta(s,\chi)$.

Let

$$\rho_0 = 1 - \lambda/\log D + it_0$$

be any zero of the function $\zeta(s,\chi)$ with $\lambda \leq \log D$, $|t_0| \leq D$. The following cases are considered separately:

$$1)\ \chi = \chi_0 \neq \chi'; \qquad 2)\ \chi \neq \chi_0 \neq \chi'; \qquad 3)\ \chi \neq \chi_0 = \chi'; \qquad 4)\ \chi = \chi_0 = \chi'. \tag{4}$$

In the first three cases the following functions, respectively, are used:

$$f(s)=\zeta(s,\chi)\zeta(s-\delta,\chi\chi'),\quad \zeta(s,\chi)\zeta(s+\delta,\chi\chi'),\quad \zeta(s,\chi)$$

and the constructions of K. A. Rodosskii ([2], Ch. II) are repeated with minor changes. In the fourth case (4) one may proceed as follows. By the process of displacement (cf. [2], Ch. I, Lemma 4) a "convenient" zero $\rho_1=\beta_1+it_1$ of the function $\zeta(s,\chi_0)$ is found. If $|\tau_1|\geq 7$, then $f(s)=\zeta(s,\chi_0)$ is used and the arguments of the third case (4) are repeated. For $|\tau_1|<7$ the function

$$f(s)=\zeta(s,\chi_0)\zeta_1(s+\delta)G(s-\delta)$$

is used, where

$$\zeta_1(s)=\zeta(s)\prod_{p/N\mathfrak{f}}(1-p^{-s}),\qquad G(s)=\zeta(s,\chi_0)/\zeta_1(s),$$

which is regular in the region $(\sigma>0,\ |t|<14)$ (where $\zeta(s)\neq 0$) and has the "convenient" zero $\rho_1+\delta$; analogous arguments are carried out with it.

Lemma 3. Let $N(\alpha,T)$ be the number of zeros of the function $\prod_\chi \zeta(s,\chi)$, belonging to the rectangle $(1-\alpha\leq\sigma\leq 1,\ |t|\leq T)$. For a suitable $C>0$ (depending only on n) and all $\lambda\in[0,\log D]$ we have

$$N\left(\lambda/\log D,\ e^{\lambda}/\log D\right)<e^{C\lambda}.$$

Let $\nu(x,\mathfrak{H})$ denote the number of ideals $\mathfrak{a}\in\mathfrak{H}$ with norm $N\mathfrak{a}\leq x$. Using (3) and contour integration, we prove the estimate

$$\nu(x,\mathfrak{H})=\gamma x+O\left(D^{2/3}x^{1-1/k}\right),\qquad \text{where } x\geq 1,\ k=\frac{1}{2}(n+3),\quad \gamma=h^{-1}\operatorname*{Res}_{s=1}\zeta(s,\chi_0), \tag{5}$$

h denotes the number of classes $\mathfrak{H}$ ($h\leq D$ according to [4], p. 66, and [9], § 3). Let

$$a_m\quad (m=1,2,\dots,X); \tag{6}$$

such a collection of ideals of the field K, for which

$$\sum_{\substack{a_m\\ \mathfrak{b}_0/a_m}}1=X/f(\mathfrak{b})+R_{\mathfrak{b}}$$

(where $\mathfrak{b}$ is any ideal and $f(\mathfrak{b})$ is a multiplicative function >0), and let N_z $(z>1)$ denote the number of those ideals $\mathfrak{b}$ which are not divisible by any prime ideal $\mathfrak{p}$ with norm $\leq z$, except for prime ideals $\mathfrak{q}$ of some set Q (by which the ideals a_m may be divisible). Further, let $\mu(\mathfrak{a})$ be the Möbius function of ideals and

$$F(\mathfrak{b}) = \sum_{\mathfrak{d}/\mathfrak{b}} \mu(\mathfrak{d}) f\left(\frac{\mathfrak{b}}{\mathfrak{d}}\right), \qquad S_z = \sum_{\substack{\mathfrak{b} \\ N\mathfrak{b} \le z \\ \mathfrak{q} \nmid \mathfrak{b}}} \frac{\mu^2(\mathfrak{b})}{F(\mathfrak{b})}, \qquad S_z(\mathfrak{i}) = \sum_{\substack{\mathfrak{b} \\ (\mathfrak{b},\mathfrak{i})=\mathfrak{o} \\ N\mathfrak{b} \le z/N\mathfrak{i},\ \mathfrak{q} \nmid \mathfrak{b}^{i}}} \frac{\mu^2(\mathfrak{b})}{F(\mathfrak{b})},$$

$$\lambda_{\mathfrak{i}} = \begin{cases} \mu(\mathfrak{i}) \prod_{\mathfrak{p}/\mathfrak{i}} (1 - 1/f(\mathfrak{p}))^{-1} S_z(\mathfrak{i})/S_z, & \text{if } \mathfrak{q} \nmid \mathfrak{i},\ N\mathfrak{i} \le z, \\ 0, & \text{otherwise.} \end{cases}$$

($\mathfrak{q} \nmid \mathfrak{b}$ means that $\mathfrak{q}$ does not divide $\mathfrak{b}$.)

By Selberg' s method ([10]) one proves the estimate

$$N_z \le X/S_z + \sum_{\substack{\mathfrak{i}_1, \mathfrak{i}_2 \\ N\mathfrak{i}_1 \le z,\ N\mathfrak{i}_2 \le z \\ \mathfrak{q} \nmid \mathfrak{i}_1 \mathfrak{i}_2}} \left| \lambda_{\mathfrak{i}_1} \lambda_{\mathfrak{i}_2} R_{\mathfrak{i}_1 \mathfrak{i}_2/(\mathfrak{i}_1, \mathfrak{i}_2)} \right|. \tag{7}$$

If in the special case (6) is the collection of all ideals $\mathfrak{a}$ of the class $\mathfrak{H}$ with $N\mathfrak{a} \le x$, then, on the basis of (5),

$$f(\mathfrak{b}) = N\mathfrak{b}, \qquad R_{\mathfrak{b}} \ll D^{3/2} (x/N\mathfrak{b})^{1-1/k}, \qquad S_z = \sum_{\in(z)} 1/N\mathfrak{b},$$

where (z) denotes the set of those ideals of the field K the product of the norms of all distinct prime ideals of which is $\le z$. Let $\mathfrak{g}$ run through all ideals of the field K divisible only by prime ideals $\mathfrak{q} \in Q$, and let $V = \sum_{\mathfrak{g}} 1/N\mathfrak{g}$; then

$$S_z \cdot V \ge \sum_{\mathfrak{b} \in (z)} 1/N\mathfrak{b} \ge \sum_{\substack{\mathfrak{b} \\ N\mathfrak{b} \le z \\ (\mathfrak{b},\mathfrak{f})=\mathfrak{o}}} 1/N\mathfrak{b}.$$

For $x \ge D^{5k}$ and $z \ge \max(x^{1/4}, D^{3k})$, the last sum is $> c_2 h\gamma \log z > c_3 h\gamma \log x$. If $\sum_{\mathfrak{q}} 1/N\mathfrak{q} \ll 1$, then $V \ll 1$ and $S_z > c_4 h\gamma \log x$, whence follows the estimate $\ll x/h \log x$ for the main term in (7). Under the additional restriction $z^{2/k} = x^{1/k}/D^{1/3}(h\gamma)^2 h \log x \ge D^6$, the same estimate can be proved for the remainder term in (7) (cf. ([11]), § 7), and under these conditions

$$N_z \ll x/h \log x. \tag{8}$$

Let $\pi(x, \mathfrak{H})$ denote the number of prime ideals of the class $\mathfrak{H}$ with norm $\le x$, and $V(x; \mathfrak{H}, y)$ the number of those ideals $\mathfrak{a}$ of the class $\mathfrak{H}$ with $N\mathfrak{a} \le x$, all prime divisors $\mathfrak{q}$ of which have norms in $[y, y^2]$ $(1 \le y \le \sqrt{x})$. Simple consequences of (8) are the estimates

$$\pi(x, \mathfrak{H}) \ll x/h \log x, \qquad V(x; \mathfrak{H}, y) \ll x/h \log x \quad (x \ge D^{5k},\ 1 \le y \le \sqrt{x}). \quad (9)$$

In the proof of the first of them the empty set Q is used (cf. ([11]), § 8).

Using the estimates (9), the lemma is proved by arguments similar to those carried out by K. A. Rodosskii ((*2*), Ch. I), with only the important difference that the numbers m, u, v ((*2*), p. 339) are replaced by the norms of such ideals $\mathfrak{a}$, all prime factors of which have norms belonging to some interval $[y, y^2]$, where $D^{\log(1+\lambda)/(2+2\lambda)} \leqslant y < D^{6k}$.

Theorem 1 is proved by the method set forth in (*12*) for the special case $n = 1$. Let q be a natural number > 1, and let χ_q be a Dirichlet character mod$q$. It can be shown that

$$\zeta(s, \chi_q, \chi) = \sum_{\mathfrak{a}} \frac{\chi_q(N\mathfrak{a})\chi(\mathfrak{a})}{N\mathfrak{a}^s} \quad (\sigma > 1) \tag{10}$$

coincides with a certain Hecke function with character modulo $\mathfrak{f}[q]$. Using the functions (10), it is proved that if there exists an ideal $\mathfrak{a} \in \mathfrak{H}$ with $N\mathfrak{a} \equiv l(\text{mod} q)$, $(l, q) = 1$, then the assertion of Theorem 1 with $D = |\Delta| q^{nN} \mathfrak{f}$ holds for primes $p \equiv l(\text{mod} q)$.

Let henceforth K be a quadratic field generated by the square root $\sqrt{\Delta}$, where Δ is the fundamental discriminant, and let $\mathfrak{f} = [k]$ with a natural number $k > 1$. Ideals $\mathfrak{a}, \mathfrak{b}$ (having no common divisor with k) are, by definition, in one and the same class $\mathfrak{C}$ if there exist integers $\alpha, \beta \in K$, congruent modulo $[k]$ to rational numbers having no common divisor with k, such that $N\alpha\beta > 0$ and $\mathfrak{a}[\alpha] = \mathfrak{b}[\beta]$. It can be shown that each class $\mathfrak{C}$ is a sum of one and the same number $j \geqslant 1$ of classes $\mathfrak{H}$. To each class $\mathfrak{C}$ there corresponds uniquely a class $\mathfrak{F}$ of primitive binary quadratic forms with discriminant $d = \Delta k^2$ (if $d < 0$, considering only positive forms), and every form $\psi \in \mathfrak{F}$ represents norms of ideals of the class $\mathfrak{C}$ ((*13*), pp. 123-124). Let $X(\mathfrak{C})$ be the characters of the classes $\mathfrak{C}$. The corresponding function $\zeta(s, X)$ is represented by the finite sum

$$\zeta(s, X) = h^{-1} \sum_{\mathfrak{C}} X(\mathfrak{C}) \sum_{\mathfrak{H} \subset \mathfrak{C}} \sum_{\chi} \overline{\chi}(\mathfrak{H})\, \zeta(s, \chi).$$

Using (3), we obtain estimates of $|\zeta(s, X)|$, which make it possible to prove Lemmas 1-3 in an analogous way with X in place of χ. For real X the function

$$\prod_{\substack{\mathfrak{p} \\ N\mathfrak{p}/\Delta,\ N\mathfrak{p} \nmid k}} (1 - X(\mathfrak{p}) N\mathfrak{p}^{-s}) \zeta(s, X)$$

is represented as the product of two Dirichlet L-functions with characters mod D ((*14*), p. 67). This fact, together with Siegel' s theorem, permits replacing in (2) the exponent by an arbitrarily small positive constant $\varepsilon > 0$ (if $D > D_0(\varepsilon)$). Using this circumstance, Theorem 2 is proved according to the scheme set forth in (*3*).

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Received
21 III 1961

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