MATHEMATICS

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Abstract

Full Text

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ISOMETRIC TRANSFORMATIONS OF PUNCTURED CONVEX SURFACES

Let F be a closed regular surface with positive Gaussian curvature. Remove from the surface F a finite number of points and denote the surface with "punctures" thus obtained by F'. The question is whether F' admits nontrivial isometric transformations. That is, do there exist regular surfaces, isometric to F', not equal to F'? The simple example of a sphere punctured at two diametrically opposite points shows that the possibility of nontrivial isometric transformations of F' is, generally speaking, not excluded. And indeed, the following theorem holds:

Theorem. *Every closed convex regular surface with positive curvature, punctured at two arbitrary points, admits at least a countable set of nontrivial isometric transformations in the class of regular surfaces.*

Proof. Define on the half-axis $\rho \geq 0$ a regular function $g(\rho)$ by the following conditions:

$$\begin{array}{lll} g'' > 0 \quad \text{for } \rho < \varepsilon; & g(0) = 0, & \displaystyle\int_0^\varepsilon g''\, d\rho = n - 1, \\ g'' = 0 \quad \text{for } \rho \geq \varepsilon; & g'(0) = 1, & \end{array}$$

where n is a positive integer. Such a function $g(\rho)$ is constructed without difficulty.

Now assign in space with cylindrical coordinates ρ, ϑ, h the metric with line element

$$ds^2 = d\rho^2 + g^2 d\vartheta^2 + dh^2.$$

We shall denote the Riemannian space thus obtained by R; it has nonpositive curvature, and in the region $R_\varepsilon : \rho > \varepsilon$ it is locally Euclidean.

Construct a special locally Euclidean space E. For this purpose introduce, in Euclidean space E_0, Cartesian coordinates x, y, z, and make in it a cut along the half-plane $y = 0,\ x \geq 0$. Take n copies of spaces with such a cut, $E_1, \dots, E_n$, and glue these spaces along the edges of the cuts E_k, E_{k+1} that are naturally adjacent, i.e. opposite. As a result of such gluing, a locally Euclidean space E with a singularity along the axis z is obtained. Denote by $E_{\varepsilon'}$ the region of the space E consisting of points at a distance greater than ε' from the axis z. With a suitable choice of $\varepsilon'(\varepsilon)$, the spaces R_ε and $E_{\varepsilon'}$ are isometric.

Now embed isometrically the surface F, which is punctured at the points A and B, in the space R in such a way that the point A coincides with the given point A_R on the axis of the space ($\rho = 0$), and the fixed two-dimensional element of the surface at this point coincides with the given element α_R of the space R at the point A_R, isometric to it. The possibility of such an isometric embedding of F in R is ensured by the theorem contained in the work ([1]). Varying

element a_R, one can arrange that the point B of the surface also falls on the axis of the space. Now let us pass to the limit as $\varepsilon \to 0$. In this process the space R passes into a locally Euclidean space E, and the surface F_R—the result of an isometric embedding of F in R—passes into a certain regular locally convex surface F_E with singularities at two points A_E and B_E, which are the limits of A_R and B_R.

Let us map the locally Euclidean space E onto the original Euclidean space E_0, assigning to an arbitrary point $X \subset E$ the point E_0 geometrically coinciding with it. This mapping is locally isometric. It takes the surface F_E into a surface F_0 isometric to it. The surface F_0, being a locally convex surface with regular metric and positive curvature, is regular everywhere except at the points A_0 and B_0, corresponding to A_E and B_E. Each ray issuing from the point C of the z-axis, situated between A_0 and B_0, intersects the surface F_0 in n points (some of them may coincide). Thus the surface F_0, isometric to the surface F punctured at the points A and B, depends essentially on the integer parameter n. It follows that a convex surface punctured at two points admits at least a countable set of nontrivial isometric transformations. The theorem is proved.

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1. A. V. Pogorelov, DAN, **137**, No. 2 (1961).

Note: Figure translations are in progress. See original paper for figures.