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Abstract

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ON THE EMBEDDING OF A k-DIMENSIONAL ELEMENT IN E^n

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The paper gives necessary and sufficient conditions for a tame embedding of a k-dimensional element in n-dimensional Euclidean space E^n (for arbitrary $k < n$) and investigates the connection between the notions of tame and normal embedding. An embedding of a k-dimensional triangulable manifold $M^k \subset E^n$ is called **tame** if there exists a homeomorphism g of the space E^n onto itself such that $g(M^k)$ is a polyhedron; an embedding $M^k \subset E^n$ is called **normal** if there exists a homeomorphism

$$h : (M^k \times I^{n-k}) \to E^n$$

(where I^{n-k} is an $(n-k)$-dimensional cube with center O) such that

$$h(M^k \times O) = M^k.$$

The first examples of "wild" (i.e. non-tame) embeddings were constructed by Antoine ([1]), Urysohn ([2]), and Aleksandrov ([3]). Criteria for tame embedding in E^3 were given by Harrold ([4]) and Bing ([5]). Brown proved that the closures of domains bounded by an $(n-1)$-dimensional sphere S^{n-1}, normally embedded in the n-dimensional sphere S^n, are n-dimensional elements ([6]).

§ 1. **Some consequences of Brown' s work.** An embedding of a continuum F in E^n (or in S^n) is called **cellular** ([6]) if there exists a system of n-dimensional elements $\{Q_i^n\}$ such that Q_{i+1}^n is contained strictly inside Q_i^n and

$$\bigcap_{i=1}^{\infty} Q_i = F.$$

Corollary 1. In order that an embedding $F \subset S^n$ be cellular, it is necessary and sufficient that $S^n \setminus F$ be homeomorphic to $S^n \setminus x$, where x is a point.

Let the embedding $F \subset S^n$ be cellular. Take in $S^n \setminus F$ a small geometric sphere S^{n-1} and denote by Q the closure of that component of $S^n \setminus S^{n-1}$ which contains F. Then, by Brown' s Theorem 1 ([6]), there exists a continuous mapping f of the element Q onto itself, identical on the boundary of Q, for which F is the unique nondegenerate inverse image of a point. Let h be a mapping identical on

$S^n \setminus Q$ and coinciding with f on $Q \setminus F$. Then h is a homeomorphism of $S^n \setminus F$ onto $S^n \setminus f(F)$, and since $f(F)$ is a point, the sufficiency is proved.

Let the embedding $F \subset S^n$ be such that $S^n \setminus F$ is homeomorphic to $S^n \setminus x$, where x is a point. Then the mapping f' of the sphere S^n (under which F is the unique nondegenerate inverse image of a point) is a mapping of this sphere onto itself; but then the embedding is cellular by Brown' s Theorem 3, and the corollary is proved.

We note that a cellular set may be wild ([7]).

Lemma 1. In order that an embedding of the $(n-1)$-dimensional sphere S^{n-1} in the n-dimensional sphere S^n be tame, it is necessary and sufficient that the closures of both components of the complement of S^{n-1} in S^n be n-dimensional elements.

Necessity is obvious. Sufficiency is not hard to prove from the fact that a homeomorphism of the sphere onto the boundary of an n-element can always be extended along radii to a homeomorphism of the ball onto the entire n-element.

In view of this same remark it is easy to prove:

Lemma 2. *If the embeddings $S^{n-1} \subset S^n$ and $S_*^{n-1} \subset S^n$ are tame and a homeomorphism $g : S^{n-1} \to S_*^{n-1}$ is given, then g can be extended to all of S^n.*

Corollary 2. *In order that an embedding $S^{n-1} \subset S^n$ be tame, it is necessary and sufficient that it be normal.**

Necessity is obvious. Sufficiency follows at once from Lemma 1 and the theorem of Brown mentioned above ([6], Theorem 5).

§ 2. Criterion for a tame embedding of a k-dimensional element in E^n.

In connection with Corollary 2 it is natural to pose the question: will a normally embedded k-dimensional element in E^n be tame? It turns out, not always: the simple arc (with one infinitely knotted end) of Artin-Fox ([7], Example 1.2) is wildly embedded in E^3, although it is not hard to show that it is normally embedded in E^3. In this example, as in any normal embedding of a k-element in E^n, wildness can arise only at boundary points; therefore it is natural to impose additional conditions on the embedding of the boundary.

Definition. A k-dimensional element Q^k is embedded in E^n **with a collar** if there exists a homeomorphism of a k-dimensional ball into E^n under which a smaller concentric ball is mapped onto Q^k.

Theorem 1. *In order that an embedding of a k-dimensional element Q^k in E^n be tame, it is necessary and sufficient that Q^k be embedded in E^n with a collar L and that the element $Q^k \cup L$ be embedded normally in E^n.*

Necessity is obvious. To prove sufficiency, consider a system of closed concentric k-dimensional balls Δ_r^k of radius r, where $0 \le r \le 2$; take an analogous system of balls Δ_s^{n-k} of dimension $n-k$ with center O and radii $0 \le s \le 1$, and consider the topological product $\Delta_2^k \times \Delta_1^{n-k}$. By the hypotheses of the theorem there exists a homeomorphism

$$h : (\Delta_2^k \times \Delta_1^{n-k}) \to E^n$$

such that

$$h(\Delta_1^k \times 0) = Q^k.$$

Consider the set

$$\partial(\Delta_{3/2}^k \times \Delta_{1/2}^{n-k}) = W_0,$$

where ∂ denotes the boundary; W_0 is an $(n-1)$-dimensional sphere bounding the topological ball

$$(\Delta_{3/2}^k \times \Delta_{1/2}^{n-k}) = Z,$$

strictly inside which lies $(\Delta_1^k \times 0)$. We show that $h(W_0)$ is normally embedded in E^n. Consider the system of topological spheres

$$W_t = \partial(\Delta_{3/2+t}^k \times \Delta_{1/2+t}^{n-k}),$$

where $-1/4 \le t \le 1/4$; $h' = h|_Z$ realizes an embedding of $h(W_0)$ in E^n with topological product by an interval, i.e. $h(W_0) \subset E^n$ is normal and, by Corollary 2, also tame. Embed

$$(\Delta_{3/2}^k \times \Delta_{1/2}^{n-k})$$

identically in E^n and extend h' to a homeomorphism h_* of all E^n onto itself. This can be done, according to Lemma 2, because the spheres W_0 and $h(W_0)$ are tame. But h_*^{-1} is a homeomorphism of E^n onto itself, and

$$h_*^{-1}(Q^k) = h^{-1}(Q^k) = (\Delta_1^k \times 0),$$

a polyhedron. The theorem is proved.

It is not hard to construct an example of a wild 2-element Q^2 which is embedded with a collar L, is embedded normally, but $Q^2 \cup L$ is not embedded normally. Let us also note that every simple arc is embedded with a collar in E^3, whereas a topological square may have no collar.

§ 3. On normally embedded k-dimensional manifolds in E^n.

It is natural to pose the question: will a normal embedding of a closed triangulable k-dimensional manifold M^k in E^n be tame? We shall show only that a normal embedding of M^k in E^n is locally tame

* This result was recently announced by Brown himself [8].

at every point (i.e., there exists a neighborhood Ox of every point $x \in M^k$ and a homeomorphism g_x such that $g_x(\overline{Ox})$ and $g_x(\overline{Ox} \cap M^k)$ are polyhedra); in the case $n = 3$ this also means that it is tame [9,10].

Theorem 2. *If the embedding in E^n of a closed triangulated k-dimensional manifold M^k is normal (or locally normal), then it is locally tame at every point.*

By hypothesis, for $x \in M^k$ there exists a k-dimensional element Q^k, $x \in Q^k \setminus \partial Q^k$, lying with its rim L in M^k, and moreover $Q^k \cup L$ is normally embedded in E^n. By Theorem 1 the embedding $Q^k \subset E^n$ will be tame; therefore there exists a homeomorphism $h(E^n) = E^n$ such that $h(Q^k) = \sigma^k$, a closed simplex. Choosing in it a smaller concentric simplex $\sigma_0^k \ni h(x)$, it is not hard to construct two open n-dimensional simplexes σ_1^n and σ_2^n, not intersecting each other and with σ_0^k, such that the open kernel V of the set $\overline{\sigma_1^n \cup \sigma_2^n}$ contains σ_0^k, and the intersection $\overline{V} \cap h(M^k)$ is equal to σ_0^k. Then $h^{-1}(V)$ will be a neighborhood of x satisfying the condition of a locally tame embedding. The theorem is proved.

For E^3 there is the Bing-Moise theorem [9,10], according to which an embedding in this space is tame if and only if it is locally tame at every point. Therefore Theorem 2 implies:

Theorem 3. *A normal embedding of a simple closed contour and of a two-dimensional closed manifold in E^3 is tame.**

For a two-dimensional manifold this assertion follows at once from the well-known Bing criterion [5].

We note that Theorem 2 is not true for manifolds with boundary, since boundary points may turn out to be locally wild. It is natural to impose additional conditions on the embedding of the boundary.

Definition. A k-dimensional manifold (with boundary) M^k is embedded in E^n **with rim** if there exists a $(k-1)$-dimensional polyhedron $\partial\mu^k$ (possibly disconnected) and a homeomorphism $h : (\partial\mu^k \times [-1, 1]) \to E^n$ such that $h(\partial\mu^k \times 0) = \partial M^k$, $h(\partial\mu^k \times (0, 1]) \cap M^k = \Lambda$, and $h(\partial\mu^k \times [-1, 0]) \subset M^k$.

By a simple modification of the proof of Theorem 2 one can prove:

Theorem 4. *If a triangulated k-dimensional manifold M^k is embedded in E^n with rim L and $M^k \cup L$ is embedded normally, then the embedding $M^k \subset E^n$ is locally tame at every point.*

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* Evidently, the converse assertion is also true.

Note: Figure translations are in progress. See original paper for figures.