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Abstract

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SELF-SIMILAR RELATIVISTIC MOTIONS IN THE CASE OF POINT SYMMETRY

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It is of interest, for the solution of certain problems, to consider all possible self-similar relativistic motions of a gas possessing point symmetry.

The basic system of equations for adiabatic motions can be written in the form

$$\frac{w}{c^2\theta^2}(u_t + uu_r) + \frac{u}{c^2}(w_t + uw_r) + \theta^2 V p_r = 0, \tag{1}$$

$$-(V_t + uV_r) + \frac{V}{\theta^2}\left(u_r + \frac{u}{c^2}u_t\right) + \frac{NVu}{r} = 0, \qquad \sigma_t + u\sigma_r = 0;$$

here

$$w = pV + \rho c^2; \qquad dw = V\,dp + T\,d\sigma; \qquad pV^k = A(\sigma);$$

w is the heat content; p is the pressure; V is the specific volume; ρc^2 is the internal volume energy; σ is the entropy (per unit total mass); T is the temperature; k is the adiabatic exponent; u is the velocity; $\theta = \sqrt{1 - u^2/c^2}$, $N = 0, 1, 2$, respectively for plane, cylindrical, and spherical waves. The last equation should be written in the form

$$w_t + uw_r - V(p_t + up_r) = 0,$$

and then the first equation takes the form

$$\frac{w}{c^2\theta^2}(u_t + uu_r) + V\left(p_r + \frac{u}{c^2}p_t\right) = 0.$$

In the classical case, for the equation of state $pV^k = A(\sigma)$, we obtained two classes of self-similar motions (^{2}), when it was assumed that

$$u = t^{a_1 - 1}\xi_1(z); \qquad \frac{1}{V} = t^{a_2}\xi_2(t); \qquad p = t^{2(a_1-1)+a_2}\xi_3(t); \qquad z = rt^{-a_1}$$

or

$$u = e^{a_1 t}\xi_1(z); \qquad \frac{1}{V} = e^{a_2 t}\xi_2(z); \qquad p = e^{(2a_1+a_2)t}\xi_3(z); \qquad z = re^{-a_1 t}.$$

In the relativistic case, as is easily verified, $a_1 = 0$ (for both variants), since the velocity of motion is limited by the speed of light.

Let us consider the case of simply relativistic flows for an equation of state which, as is known, is valid for them as well, and separately the case of ultrarelativistic flows, when $p = (k-1)\rho c^2$.

In the case of relativistic flows, for the first class of self-similar motions we have

$$u = u(z) = \xi_1(z); \qquad \frac{1}{V} = t^{a_2}\xi_2(z); \qquad p = t^{a_2}\xi_3(z); \qquad z = \frac{r}{t},$$

where

$$w = \frac{k}{k-1}pV + c^2 = \frac{k}{k-1}\frac{\xi_3}{\xi_2} + c^2 = w(z).$$

Substitution of these quantities into the basic system of equations leads to new equations:

$$(u-z)\left(\frac{k}{k-1} + \frac{\xi_2}{\xi_3}c^2\right)\frac{u'}{c^2\theta^2} + \frac{\xi_3'}{\xi_3}\left(1 - \frac{uz}{c^2}\right) + \frac{a_2 u}{c^2} = 0,$$

$$a_2 + (u-z)\frac{\xi_3'}{\xi_3} + \frac{u'}{\theta^2}\left(1 - \frac{uz}{c^2}\right) + \frac{Nu}{z} = 0,$$

$$(u-z)\left[-\frac{\xi_3'}{\xi_3} + k\frac{\xi_2'}{\xi_2}\right] + (k-1)a_2 = 0;$$

here, for example, $u' = du/dz$.

Relations analogous to these were obtained earlier from dimensional-theory considerations (^{3}).

Introduce $\eta = \xi_2/\xi_3 = 1/pV$; then the last system of equations takes the form:

$$\left(\frac{k}{k-1} + c^2\eta\right)\frac{u'}{c^2\theta^2}(u-z) + \left(\frac{\xi_2'}{\xi_2} - \frac{\eta'}{\eta}\right)\left(1 - \frac{uz}{c^2}\right) + \frac{a_2 u}{c^2} = 0,$$

$$a_2 + (u-z)\frac{\xi_2'}{\xi_2} + \frac{u'}{\theta^2}\left(1 - \frac{uz}{c^2}\right) + \frac{Nu}{z} = 0,$$

$$(u-z)\left[(k-1)\frac{\xi_2'}{\xi_2}+\frac{\eta'}{\eta}\right]+(k-1)a_2=0.$$

Hence it follows that

$$\frac{\eta'}{\eta}=\frac{k-1}{u-z}\left[\frac{u'}{\theta^2}\left(1-\frac{uz}{c^2}\right)+\frac{Nu}{z}\right],$$

$$\frac{\xi_2'}{\xi_2}=-\frac{1}{u-z}\left[a_2+\frac{u'}{\theta^2}\left(1-\frac{uz}{c^2}\right)+\frac{Nu}{z}\right],$$

$$\left(\frac{k}{k-1}+c^2\eta\right)\frac{u'}{c^2\theta^2}(u-z)-\frac{1-uz/c^2}{u-z}\left[a_2+\frac{ku'}{\theta^2}\left(1-\frac{uz}{c^2}\right)+\frac{kNu}{z}\right]+\frac{a_2u}{c^2}=0.$$

Eliminating η from the last two equations, one can arrive at a single second-order equation for determining u, after which ξ is found by quadrature, and η is determined algebraically. In special cases of motions the problem reduces to the solution of first-order equations; sometimes exact integrals can be found.

Let us investigate the second class of motions. In this case $u=u(r)=\xi_1(r)$; $\frac{1}{V}=e^{a_2t}\xi_2(r)$; $p=e^{a_2t}\xi_3(r)$; $z=r$. Substituting these quantities into the basic equations, we arrive at the expressions:

$$\left(\frac{k}{k-1}+c^2\frac{\xi_2}{\xi_3}\right)\frac{uu'}{c^3\theta^2}+\frac{\xi_3'}{\xi_3}+\frac{a_2u}{c^2}=0,$$

$$a_2+u\frac{\xi_2'}{\xi_2}+\frac{u'}{\theta^2}+\frac{Nu}{r}=0,$$

$$u\left(-\frac{\xi_3'}{\xi_3}+k\frac{\xi_2'}{\xi_2}\right)+(k-1)a_2=0$$

(here, for example, $u'=du/dr$).

Introduce $\eta=\xi_2/\xi_3=1/pV$; then this system of equations takes the form

$$\left(\frac{k}{k-1}+c^2\eta\right)\frac{uu'}{c^2\theta^2}+\left(\frac{\xi_2'}{\xi_2}-\frac{\eta'}{\eta}\right)+\frac{a_2u}{c^2}=0;$$

$$a_2+u\frac{\xi_2'}{\xi_2}+\frac{u'}{\theta^2}+\frac{Nu}{r}=0;$$

$$u\left[(k-1)\frac{\xi_2'}{\xi_2}+\frac{\eta'}{\eta}\right]+(k-1)a_2=0.$$

Hence it follows that

$$\frac{\xi_2'}{\xi_2}=-\left[\frac{a_2}{u}+\frac{u'}{u\theta^2}+\frac{N}{r}\right],$$

$$\frac{\eta'}{\eta}=(k-1)\left[\frac{u'}{u\theta^2}+\frac{N}{r}\right],$$

$$\left(\frac{k}{k-1}+\eta c^2\right)\frac{uu'}{c^2\theta^2}-\left(\frac{a_2}{u}+\frac{ku'}{u\theta^2}+\frac{kN}{r}\right)+\frac{a_2u}{c^2}=0.$$

It is convenient to write the last equation in the form

$$\left(\frac{k}{k-1}+c^2\eta\right)\frac{uu'}{c^2\theta^2}-k\left(\frac{u'}{u\theta^2}+\frac{N}{r}\right)-\frac{a_2\theta^2}{u}=0. \tag{1}$$

The second equation of this system admits the obvious integral

$$\eta=\frac{A_1}{c^2}\,\frac{(ur^N)^{k-1}}{(1-u^2/c^2)^{\frac{k-1}{2}}}=\frac{A_1}{c^2}\left(\frac{ur^N}{\theta}\right)^{k-1},$$

where A_1 is a constant of integration, after which equation (1) can be written in the form

$$\frac{k}{k-1}+A_1\left(\frac{ur^N}{\theta}\right)^{k-1}\frac{uu'}{c^2\theta^2}-k\left(\frac{u'}{u\theta^2}+\frac{N}{r}\right)+\frac{a_2\theta^2}{u}=0.$$

Thus, the problem has been reduced to the solution of one first-order equation and two quadratures, one of which is immediately taken in finite form (the law of conservation of mass flux).

In special cases of motions one can seek exact systems. It should be noted that the first class of solutions is obtained as depending on 5 constants: a_2, A_1, A_2, A_3 (constants of integration) and τ, since the time t is determined up to an arbitrary constant τ. In the classical case we obtained solutions depending on 6 constants, since in that case $a_1 \neq 0$, which formally gave a complete integral (we note that the use of the constant τ has practically no great meaning).

The second class of solutions contains only 4 constants, since the constant τ introduces only constant multipliers for p and V, which cancel in the determination of pV and w.

Let us pass to the case of ultrarelativistic motions. In this case

$$w = \frac{k}{k-1} pV.$$

The basic system of equations can then be written in the form

$$\frac{k}{(k-1)c^2\theta^2}(u_t + uu_r) + (\ln p)_r + \frac{u}{c^2}(\ln p)_t = 0,$$

$$-\left[(\ln V)_t + u(\ln V)_r\right] + \frac{1}{\theta^2}\left(u_r + \frac{u}{c^2}u_t\right) + \frac{Nu}{r} = 0,$$

$$(\ln p)_t + u(\ln p)_r + k\left[(\ln V)_t + u(\ln V)_r\right] = 0.$$

which makes it possible to eliminate $(\ln V)_t + u(\ln V)_r$ from the second equation of this system and write it in the form

$$\frac{1}{c^2\theta^2}(u_t + uu_r) + p^*_r + \frac{u}{c^2}p^*_t = 0,$$

$$p^*_t + up^*_r + \frac{k-1}{\theta^2}\left(u_r + \frac{u}{c^2}u_t\right) + \frac{(k-1)Nu}{r} = 0, \qquad (2)$$

$$(\ln V)_t + u(\ln V)_r + \frac{1}{k-1}(p^*_t + up^*_r) = 0, \qquad p^* = \frac{k-1}{k}\ln p.$$

For the first class of self-similar motions one must again set $a_1 = 0$; in this case $u = u(z) = \xi_1(z)$, but it is then possible to introduce two arbitrary constants by assuming that $1/V = t^{a_2}\xi_2(z)$, $p = t^{a_3}\xi_3(z)$; then $p^* = \ln(t^{a_3}\xi_3)^{\frac{k-1}{k}}$. In this case system (2) takes the form

$$\frac{k}{k-1}\frac{u'}{c^2\theta^2}(u-z) + \frac{\xi_3'}{\xi_3}\left(1 - \frac{uz}{c^2}\right) + \frac{ua_3}{c^2} = 0,$$

$$a_3 + (u-z)\frac{\xi_3'}{\xi_3} + \frac{k}{\theta^2}\left(1 - \frac{uz}{c^2}\right) + \frac{kNu}{z} = 0,$$

$$k\left[a_2 + (u-z)\frac{\xi_2'}{\xi_2}\right] + a_3 + (u-z)\frac{\xi_3'}{\xi_3} = 0.$$

Eliminating from the first two equations the quantity ξ_3'/ξ_3, we arrive at a first-order equation determining $u = u(z)$:

$$\frac{k}{k-1}\frac{u'}{c^2\theta^2}(u-z)-\left(1-\frac{uz}{c^2}\right)\frac{1}{(u-z)}\left[a_3+\frac{ku'}{\theta^2}\left(1-\frac{uz}{c^2}\right)+\frac{kNu}{z}\right]+\frac{a_3u}{z}=0,$$

after which ξ_2 and ξ_3 are determined by quadratures.

In the case of the second class of self-similar motions we assume that $u = u(r) = \xi_1(r)$, $1/V = e^{a_2 t}\xi_2(r)$, $p = e^{a_3 t}\xi_3(r)$; then

$$p^* = \ln(e^{a_3 t}\xi_3)^{\frac{k-1}{k}} = \frac{k-1}{k}(a_3 t + \ln \xi_3).$$

Equations (2) then take the form:

$$\frac{k}{k-1}\frac{u'u}{c^2\theta^2}+\frac{\xi_3'}{\xi_3}+\frac{a_3u}{c^2}=0, \qquad a_3+u\frac{\xi_3'}{\xi_3}+\frac{ku'}{\theta^2}+\frac{kNu}{r}=0,$$

$$k\left(a_2+u\frac{\xi_2'}{\xi_2}\right)+a_3+u\frac{\xi_3'}{\xi_3}=0.$$

Eliminating from the first two equations the quantity ξ_3'/ξ_3, we arrive at the first-order equation determining $u = u(z)$:

$$\frac{k}{k-1}\frac{uu'}{c^2\theta^2}-\left(\frac{a_3}{u}+\frac{ku'}{u\theta^2}+\frac{kN}{r}\right)+\frac{a_3u}{c^2}=0.$$

Further, ξ_2 and ξ_3 are determined by quadratures. For the first class of self-similar motions we formally obtain solutions of the complete-integral type, since this solution contains 6 arbitrary constants: $a_2, a_3, A_1, A_2, A_3, \tau$.

The second class contains only 5 constants: a_2, a_3, A_1, A_2, A_3; the constant τ has a trivial meaning—it introduces only constant factors for p and V; since p and V enter the equation logarithmically, these factors cancel.

In the case of motion of a relativistic gas with velocity close to the speed of light, one can find asymptotic general solutions which, in a certain sense, will also be self-similar. Let $u/c = 1 - 2\Delta$; then $\theta^2 = 4\Delta(1-\Delta)$, and the basic system of equations can be written in the following form:

$$-\frac{w}{2\Delta(1-\Delta)V}\left[\Delta_\tau+(1-2\Delta)\Delta_r\right]+p_\tau+(1-2\Delta)p_r=0,$$

$$(\ln V)_\tau+(1-2\Delta)(\ln V)_r+\frac{1}{2\Delta(1-\Delta)}\left[\Delta_r+(1-2\Delta)\Delta_\tau\right]-\frac{N}{r}(1-2\Delta)=0,$$

$$\sigma_\tau + (1 - 2\Delta)\sigma_r = 0, \qquad \tau = ct.$$

If in some frame of reference $\Delta \ll 1$, then this system of equations takes the form:

$$\frac{1}{2\Delta}(\Delta_\tau + \Delta_r) = \frac{V}{w}(p_\tau + p_r),$$

$$\ln\left(\Delta^{1/2} V r^{-N}\right)_\tau + \ln\left(\Delta^{1/2} V r^{-N}\right)_r = 0,$$

$$\sigma_\tau + \sigma_r = 0.$$

It follows hence that $\sigma = \sigma(r - \tau)$, therefore

$$pV^k = A(\sigma) =$$

$= f_1(r - \tau)$. The first equation of the system can now be written in the form:

$$\frac{1}{2\Delta}(\Delta_\tau + \Delta_r) = \frac{1}{w}(w_r + w_t), \qquad \frac{w}{2V\sqrt{\Delta}} = \frac{w}{\theta} = f_2(r - \tau), \qquad \frac{Br^N c}{V\theta} = \frac{Br^N a}{V\theta} = f_3(r - \tau),$$

where $B = 1, 2\pi, 4\pi$ for $N = 0, 1, 2$. The solutions of these equations express the conservation laws for entropy (energy), the integral of the quantity of motion (energy flux), and mass.

In the case of the corresponding stationary motions we have (6):

$$pV^k = f_{10} = \text{const}, \qquad w/\theta = f_{20} = \text{const}, \qquad Br^{Na}/V\theta = f_{30} = \text{const}.$$

In the general case $w = \frac{k}{k-1} pV + ac^2$, where $\alpha = 1$ in the relativistic case and $\alpha = 0$ in the ultrarelativistic case. Eliminating w, V, and θ from these equations, we arrive at the equation determining $p = p(r - \tau)$:

$$\left(\frac{k}{k-1} f_1^{\frac{1}{k}} p^{\frac{k-1}{k}} + \alpha c^2\right) = \frac{Bcf_2}{f_3 f_1^{\frac{1}{k}}} p^{\frac{1}{k}} r^N.$$

Next one can determine $V = V(r - \tau)$ and $a = a(r - \tau)$.

In the case of an ultrarelativistic gas $\alpha = 0$, and we shall have

$$p = F_1(r-\tau)r^{-\frac{kN}{2-k}}, \qquad V = F_2(r-\tau)r^{\frac{N}{2-k}}, \qquad \theta = F_3(r-\tau)r^{-\frac{N(k-1)}{2-k}}.$$

In the study of ultrarelativistic flows, the value $k = 4/3$ is meaningful. For $1 < k < 4/3$ the equation approximately describes the motion of a relativistic gas, since the equation $w = \dfrac{k}{k-1}pV + c^2$ can be approximately approximated by the equation $w = \dfrac{k^*}{k^*-1}$, where k^* is the "effective" adiabatic exponent, with $k^* < k$. For $k = 4/3$ we have $p = F_1 r^{-2N}$, $V = F_2 r^{3/2N}$, $\theta = F_3 r^{-N/r}$. For example, for $N = 2$, $p = F_1 r^{-4}$, $V = F_2 r^3$, $\theta = F_3 r^{-1}$.

The relations obtained may be used in the theory of propagation of ultrarelativistic waves and, in particular, shock waves.

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Note: Figure translations are in progress. See original paper for figures.