

# ON THE SOLUTION OF BOUNDARY-VALUE PROBLEMS WITH COMPLEX BOUNDARY CONDITIONS BY MEANS OF SIMPLE BOUNDARY-VALUE PROBLEMS

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**Abstract**

**Full Text**

**MATHEMATICS**

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**ON THE SOLUTION OF BOUNDARY-VALUE PROBLEMS WITH COMPLEX BOUNDARY CONDITIONS BY MEANS OF SIMPLE BOUNDARY-VALUE PROBLEMS**

*(Presented by Academician A. A. Dorodnitsyn, November 29, 1960)*

We shall consider the problem of integrating a system of linear differential equations of order  $n$

$$\frac{dx}{dt} = A(t)x + f(x) \tag{1}$$

with the following boundary conditions:

$$Bx(0) + Cx(l) = d \tag{2}$$

(problem I). Here  $A(t)$  is the matrix of coefficients of the system, the elements of which are integrable functions;  $B, C$  are constant square matrices of order  $n$ ;  $d$  is a given vector,  $x(t)$  is the sought vector-function. The solution of the problem is sought on the interval  $[0, l]$ .

Representing  $x(t)$  in the form of the sum  $x(t) = y(t) + z(t)$ , where  $y(t)$  is the solution of the boundary-value problem

$$\frac{dy}{dt} = A(t)y + f(t), \quad \tilde{B}y(0) + \tilde{C}y(l) = 0, \tag{3}$$

we reduce problem I in the usual way to a problem with a homogeneous system of equations

$$\frac{dz}{dt} = A(t)z, \quad Bz(0) + Cz(l) = d - \tilde{B}y(0) - \tilde{C}y(l) = g \tag{4}$$

(problem I).

Let us also consider, alongside problem (4), a boundary-value problem of the form

$$\frac{dz}{dt} = A(t)z, \quad \widetilde{B}z(0) + \widetilde{C}z(l) = u \quad (5)$$

(problem II), where  $\widetilde{B}, \widetilde{C}$  are given constant square matrices.

For each solution of problem I one can find such a vector  $u$  that problem II will also be solvable and one of its solutions will coincide with the solution of problem I. Let us additionally assume that problem II has a unique solution for any  $u$ .

Suppose that the solution of problem II can be obtained in a simpler way than the solution of problem I. Then one can construct an iterative process for solving problem I, at each step of which it is necessary to solve only boundary-value problems of type II. For definiteness, in what follows we shall carry out the exposition for the case when problem II is the simplest one, i.e. when as the boundary conditions specify the first  $m$  components of the vector  $z(0)$  and the first  $n - m$  components of the vector  $z(l)$ . In this case the matrices  $\widetilde{B}$  and  $\widetilde{C}$  take the form:

$$\widetilde{B} = \left( \begin{array}{cccc} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 \end{array} \right) \left. \vphantom{\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}} \right\} m \text{ rows}, \quad \widetilde{C} = \left( \begin{array}{cccc} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 \end{array} \right) \left. \vphantom{\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array}} \right\} n-m \text{ rows}.$$

For  $m = n$ , boundary-value problem II becomes a Cauchy problem. We note that the case in which  $m$  components of the vector  $z(0)$  and  $n - m$  components of the vector  $z(l)$  are specified in an arbitrary order is treated in a completely analogous way.

Let  $\Phi(t)$  denote the fundamental matrix of solutions of system (1), i.e. the functional matrix satisfying the following conditions:

$$\frac{d\Phi}{dt} = A(t)\Phi, \quad \Phi(0) = E,$$

where  $E$  is the identity matrix. Let  $\Phi(l) = \Phi_l$ . Then  $z(t) = \Phi(t)z(0)$ , and from (4) and (5) we obtain a linear algebraic equation for the vector  $u$ :

$$(B + C\Phi_l)(\widetilde{B} + \widetilde{C}\Phi_l)^{-1}u = g. \quad (6)$$

To solve equation (6), one may use those iterative processes which do not require splitting the matrix  $\Phi_l$  into parts <sup>(1,2)</sup>. In this case there is no need to determine the matrix  $\Phi_l$  in advance. We shall apply here the iterative process described in <sup>(3)</sup>.

**Computational scheme of the process.** Specify arbitrarily a vector  $u_0$  and find the solution of the boundary-value problem:

$$\frac{d\xi_0}{dt} = A(t)\xi_0, \quad \widetilde{B}\xi_0(0) + \widetilde{C}\xi_0(l) = u_0$$

We compute the residual

$$r_0 = B\xi_0(0) + C\xi_0(l).$$

Denoting transposed matrices by an asterisk above, introduce the vector

$$p_0 = \{e_{n-m+1}^{(0)}, e_{n-m+2}^{(0)}, \dots, e_n^{(0)}, f_{m+1}^{(0)}, f_{m+2}^{(0)}, \dots, f_n^{(0)}\},$$

where  $e_i^{(0)}, f_i^{(0)}$  are the components of the vectors  $C^*r_0, B^*r_0$ :

$$C^*r_0 = \{e_1^{(0)}, e_2^{(0)}, \dots, e_n^{(0)}\}, \quad B^*r_0 = \{f_1^{(0)}, f_2^{(0)}, \dots, f_n^{(0)}\}.$$

We also introduce the matrices

$$B' = \left. \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 & 1 \end{pmatrix} \right\} n-m \text{ rows}, \quad C' = \left. \begin{pmatrix} 0 & 0 & \dots & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 0 & \dots & 0 & 0 \end{pmatrix} \right\} m \text{ rows}.$$

We now find the solution of the boundary-value problem:

$$\frac{d\eta_0}{dt} = -A^*(t)\eta_0, \quad B'\eta_0(0) + C'\eta_0(l) = p_0.$$

This boundary-value problem is of the type of problem II, i.e., its boundary condition also consists in the fact that part of the components of the sought vector-function  $\eta_0(t)$  is prescribed at one end of the interval of integration, and the remaining part at the other. The subsequent computations are carried out according to the formulas:

$$\widetilde{C}^*\zeta_0 = C^*r_0 + \eta_0(l), \quad \widetilde{B}^*\zeta_0 = B^*r_0 - \eta_0(0),$$

which make it possible to determine all components of the vector  $\zeta_0$ . Before the beginning of the main cycle of the process we also determine  $d_0 = (r_0, r_0)$ , where the scalar product of vectors is denoted by round brackets, and set  $b_{-1} = z_{-1} = 0$ .

The main cycle of the process, for any  $k = 0, 1, 2, \dots$ , is constructed in the following way. We compute

$$c_k = \frac{d_k}{(\zeta_k, \zeta_k)}, \quad u_{k+1} = u_k - c_k \zeta_k$$

and solve the boundary-value problem

$$\frac{d\xi_k}{dt} = A(t)\xi_k, \quad \widetilde{B}\xi_k(0) + \widetilde{C}\xi_k(l) = \zeta_k.$$

Hence

$$\chi_k = B\xi_k(0) + C\xi_k(l).$$

We find the solution of the boundary-value problem for the adjoint system

$$\frac{d\eta_k}{dt} = -A^*(t)\eta_k, \quad B'\eta_k(0) + C'\eta_k(l) = p_k,$$

where  $p_k$  is the vector constructed analogously to the vector  $p_0$ :

$$p_k = \{e_{n-m+1}^{(k)}, e_{n-m+2}^{(k)}, \dots, e_n^{(k)}, f_{m+1}^{(k)}, f_{m+2}^{(k)}, \dots, f_n^{(k)}\},$$

$$C^*\chi_k = \{e_1^{(k)}, e_2^{(k)}, \dots, e_n^{(k)}\}, \quad B^*\chi_k = \{f_1^{(k)}, f_2^{(k)}, \dots, f_n^{(k)}\}.$$

We then perform the following computations:

$$\widetilde{C}^*\omega_k = C^*\chi_k + \eta_k(l), \quad \widetilde{B}^*\omega_k = B^*\chi_k - \eta_k(0),$$

$$a_k = \frac{(\omega_k, \zeta_k)}{(\zeta_k, \zeta_k)}, \quad b_{k-1} = \frac{(\zeta_k, \zeta_k)}{(\zeta_{k-1}, \zeta_{k-1})},$$

$$\zeta_{k+1} = \omega_k - a_k \zeta_k - b_{k-1} \zeta_{k-1}.$$

To compute the coefficient  $d_k$  we use the relations

$$d_1 = (\zeta_0, \zeta_0) - a_0 d_0, \quad d_{k+1} = -a_k d_k - b_{k-1} d_{k-1} \quad (k = 1, 2, \dots).$$

The process continues until  $\|u_{k+1} - u_k\| = \sqrt{(u_{k+1} - u_k, u_{k+1} - u_k)}$  becomes sufficiently small. Then the solution of the original problem I is obtained as the solution of the problem of type II:

$$\frac{dx}{dt} = A(t)x + f(t), \quad \widetilde{B}x(0) + \widetilde{C}x(l) = u, \quad (7)$$

where  $u = \lim u_k$ .

**Questions of convergence.** The process described, if computational error is not taken into account, must lead to the exact solution in a finite number of steps not exceeding  $n$ . Among processes of a fairly general form

$$u_k = P_k(S)u_0 + R_k(S)g,$$

where  $P_k(S), R_k(S)$  are polynomials in the matrix

$$S = [(B + C\Phi_l)(\widetilde{B} + \widetilde{C}\Phi_l)^{-1}]^* [(B + C\Phi_l)(\widetilde{B} + \widetilde{C}\Phi_l)^{-1}],$$

the process under consideration gives, for each  $k$ , the smallest value of the quantity  $\|u_k - u\|^{(3)}$ , i.e., from the point of view of the Hilbert norm it is the process with the best convergence.

In constructing the computational scheme of the iterative process, the following lemma was used.

**Lemma.** If  $\tau_1, \tau_2$  are two arbitrary points of the interval  $[0, l]$ , and  $\Phi(t)$  is such a fundamental matrix of the system  $dx/dt = A(t)x$  that  $\Phi(\tau_1) = E$ , then the fundamental matrix  $\Psi(t)$  ( $\Psi(\tau_2) = E$ ) of the system  $dx/dt = -A^*(t)x$  satisfies the condition  $\Psi(\tau_1) = \Phi^*(\tau_2)$ .

To formulate the convergence theorem for the process, we introduce the homogeneous adjoint boundary-value problem

$$\frac{dx}{dt} = -A^*(t)x, \quad x(0) = -B^*v, \quad x(l) = C^*v. \quad (8)$$

Denote by  $D$  the subspace of vectors  $v$  satisfying the conditions (8), and by  $\tilde{g}$  the projection of the vector  $g$  onto the subspace orthogonal to  $D$ .

**Theorem 1.** The sequence of vectors  $u_k$ , computed by means of the iterative process described above, converges for an arbitrary choice of the initial vector  $u_0$ , and moreover:

- a) if boundary-value problem I is solvable (possibly nonuniquely), then the convergence will be to such a vector  $u$  for which the solution of boundary-value problem (7) coincides with one of the solutions of problem I;
- b) if boundary-value problem I is unsolvable, then the convergence will be to a vector  $u$  for which the solution of boundary-value problem (7) gives a solution of a problem of type I, but with such a vector  $d$  for which  $d - By(0) - Cy(l) = \tilde{g}$ .

We note that in special cases, when boundary-value problem I has a nonunique solution or is unsolvable, the iterative process accumulates computational errors and, for large  $n$ , does not always lead to the required result.

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*Note: Figure translations are in progress. See original paper for figures.*

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