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Abstract

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ON SOME QUASILINEAR PARABOLIC SYSTEMS WITH GROWING COEFFICIENTS

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In paper (1) the existence of a solution of the first boundary-value problem and of the Cauchy problem in the large was proved for quasilinear parabolic systems of the form

$$\begin{gathered}\frac{\partial^2 u_i}{\partial x^2} = \frac{\partial u_i}{\partial t} + \sum b_{ij}(x,t,u)\frac{\partial u_j}{\partial x} + f_i(x,t,u),\\ u=(u_1,\dots,u_N),\qquad i=1,\dots,N,\end{gathered}\tag{1}$$

with bounded coefficients b_{ij} (and under certain assumptions concerning the functions f_i).

Set

$$B(M) = \max_{|u_i|\le M} |b_{ij}(x,t,u)|.$$

A small modification of the proof of the existence theorem in (1) makes it possible to prove the existence of a solution of the first boundary-value problem and of the Cauchy problem in the large when

$$\int^{\infty} \frac{dM}{MB^2(M)} = \infty.$$

Analogous theorems for the Cauchy problem were proved in the paper of S. D. Eidel’ man (2).

In the present paper systems of the form

$$\begin{gathered}\varepsilon\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + \frac{\partial \varphi(u,v)}{\partial x},\\ \varepsilon\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} + \frac{\partial \psi(u,v)}{\partial x}\end{gathered}\tag{2}$$

are considered.

For such systems, in some cases it is possible to prove existence theorems in the large even when the coefficients of $\partial u/\partial x$ and $\partial v/\partial x$ have more rapid growth in u and v.

In paper (^{1}) (Theorem 4) it was proved that the solution of the first boundary-value problem for any system of the form (2) with sufficiently smooth φ, ψ and boundary functions exists for all values of t, if there is an a priori estimate for u and v in C. In the present paper, for solutions of certain systems of the form (2), an a priori estimate of a different character is established, and it is then indicated for which systems boundedness of u and v in C follows from this estimate.

For system (2) the first boundary-value problem is considered with the conditions

$$u\big|_{t=0} = u_0(x), \qquad v\big|_{t=0} = v_0(x), \tag{3}$$

$$u\big|_{x=x_1} = u\big|_{x=x_2} = 0, \qquad v\big|_{x=x_1} = v\big|_{x=x_2} = 0.$$

Everywhere in what follows it is assumed that the first-order system corresponding to (2) as $\varepsilon \to 0$ is of hyperbolic type.

Consider the equation

$$\varphi_v F_{uu} - (\varphi_u - \psi_v) F_{uv} - \psi_u F_{vv} = 0. \tag{4}$$

This equation has the same type as system (2) for $\varepsilon = 0$. Indeed, the type of equation (4) is determined by the sign of the expression

$$(\varphi_u - \psi_v)^2 + 4\varphi_v \psi_u. \tag{5}$$

and the type of the system by the roots of the equation

$$\lambda^2 - \lambda(\varphi_u + \psi_v) + (\varphi_u \psi_v - \psi_u \varphi_v) = 0,$$

whose discriminant coincides with (5).

Theorem 1. *If equation (4) has a solution $F(u, v)$ such that, for all u, v,*

$$F_{uu}\xi^2 + 2F_{uv}\xi\eta + F_{vv}\eta^2 \geq \mu(u, v)(\xi^2 + \eta^2), \qquad \mu > 0, \tag{6}$$

then

$$\int_{x_1}^{x_2} F(u(x,T), v(x,T))\, dx \le \int_{x_1}^{x_2} F(u_0(x), v_0(x))\, dx. \tag{7}$$

Proof. Multiply the equations of system (2) by F_u, F_v, respectively, add, and integrate over the rectangle $R\{x_1 \le x \le x_2,\ 0 \le t \le T\}$; the integral over R will be denoted by $[\cdot]$. We have

$$\varepsilon \left[F_u \frac{\partial^2 u}{\partial x^2} + F_v \frac{\partial^2 v}{\partial x^2} \right] = \left[\frac{\partial F}{\partial t} \right] + \left[F_u \left(\varphi_u \frac{\partial u}{\partial x} + \varphi_v \frac{\partial v}{\partial x} \right) + F_v \left(\psi_u \frac{\partial u}{\partial x} + \psi_v \frac{\partial v}{\partial x} \right) \right]. \tag{8}$$

Since the function $F(u,v)$ is a solution of equation (4), the last term in (8) has the form $\partial G(u,v)/\partial x$.

By adding to $F(u,v)$ a suitably chosen linear function, one can ensure that the conditions

$$F(0,0) = F_u(0,0) = F_v(0,0) = 0 \tag{9}$$

are satisfied.

Integration by parts gives

$$-\varepsilon \left[F_{uu} \left(\frac{\partial u}{\partial x} \right)^2 + 2F_{uv} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + F_{vv} \left(\frac{\partial v}{\partial x} \right)^2 \right] = \int_{x_1}^{x_2} F\Big|_{t=T} dx - \int_{x_1}^{x_2} F\Big|_{t=0} dx. \tag{10}$$

(The integral of $\partial G/\partial x$ vanishes by virtue of the boundary conditions.) Since the function F is convex, (7) follows from (10).

Introduce the notation:

$$B(M) = \max_{\substack{x,t \in R \\ |u|,\,|v| \le M}} \left(|\varphi_u|,\ |\varphi_v|,\ |\psi_u|,\ |\psi_v| \right), \qquad B_1(M) = \max_{\substack{x,t \in R \\ |u|,\,|v| \le M}} \left(|D^2\varphi|,\ |D^2\psi| \right)$$

(D^2 is any second derivative),

$$f(|u|) = \min \left(\min_v F(u,v),\ \min_v F(-u,v) \right),$$

$$g(|v|) = \min \left(\min_u F(u,v),\ \min_u F(u,-v) \right).$$

Theorem 2. *Let the coefficients of system (2) and the boundary functions satisfy the smoothness conditions formulated in Theorem 1 of paper (1), and let equation (4) have a solution $F(u,v)$ satisfying conditions (6), (9) and such that*

$$B(M) + MB_1(M) = o\left(f\left(\frac{M}{2}\right)\right),$$

$$B(M) + MB_1(M) = o\left(g\left(\frac{M}{2}\right)\right). \tag{11}$$

Then the solution of the first boundary-value problem (2), (3) exists for all t.

Proof. As was already stated, to prove the existence theorem it is enough to obtain an a priori estimate in C for u and v. Let

$$M = \max_{x,t\in R}(|u|,|v|), \qquad M_1 = \max_{x,t\in R}\left(\left|\frac{\partial u}{\partial x}\right|, \left|\frac{\partial v}{\partial x}\right|\right).$$

From the estimates ob-

of [1] (they are set out in detail in [3]), it follows that

$$M_1 \leq K(MB(M) + M^2B_1(M)). \tag{12}$$

Let now, for definiteness, $M = \max|u|$. From estimate (7) it follows that

$$\int_{x_1}^{x_2} f(|u|)\big|_{t=T}\,dx \leq \int_{x_1}^{x_2} F(u_0(x), v_0(x))\,dx = A. \tag{13}$$

Let E be the set on the interval $[x_1, x_2]$ where $M/2 \leq |u(x,T)| \leq M$. By virtue of (13), $\operatorname{mes} E \leq A/f(M/2)$, whence it follows that

$$M_1 \geq \max\left|\frac{\partial u}{\partial x}\right| \geq \frac{Mf(M/2)}{A}. \tag{14}$$

The estimates (12), (14) give

$$\frac{Mf(M/2)}{A} \leq M_1 \leq K(MB(M) + M^2B_1(M)). \tag{15}$$

If $M = \max|v|$, we similarly obtain

$$\frac{Mg(M/2)}{A} \leq M_1 \leq K(MB(M) + M^2B_1(M)). \tag{16}$$

From condition (11) and either of these estimates, the boundedness of M follows.

We shall say that a function $\varphi(s)$ grows like $|s|^p$ if

$$k_1|s|^p \leq |\varphi(s)| \leq k_2(1+|s|^p).$$

Theorem 3. Let $\varphi = \varphi(v)$, $\psi = \psi(u)$, and suppose that $\varphi(s)$ and $\psi(s)$ grow like $|s|^p$, and that the order of growth is lowered by 1 under differentiation and raised by 1 under integration of these functions. Then, if the coefficients of system (2) and the boundary functions (3) satisfy the smoothness conditions of Theorem 1 of [1], the solution of problem (2), (3) exists for all t.

Proof. Equation (4) in this case has the form

$$\varphi'(v)F_{uu} = \psi'(u)F_{vv},$$

and a convex solution of it is the function

$$F(u,v) = \int_0^u \psi(u)\,du + \int_0^v \varphi(v)\,dv.$$

Indeed, by virtue of the hyperbolicity of (2) for $\varepsilon = 0$, $\varphi'(v)$ and $\psi'(u)$ have the same sign, and without loss of generality they may be regarded as positive. Conditions (9) are satisfied, since one can always assume that $\varphi'(0) = \psi'(0) = 0$.

We have

$$f(|u|) = \min\left(\int_0^u \psi(u)\,du,\ \int_{-u}^0 \psi(u)\,du\right),$$

$$g(|v|) = \min\left(\int_0^v \varphi(v)\,dv,\ \int_{-v}^0 \varphi(v)\,dv\right),$$

i.e. $f(M)$ and $g(M)$ grow like M^{p+1}. The function $F(M)+MB_1(M)$ grows like M^{p-1}, whence estimate (11) follows. Consequently, problem (2), (3) satisfies the conditions of Theorem 2 and has a solution for all t.

We note that estimate (7) does not depend on ε. The estimate for $M = \max(|u|,|v|)$, which is obtained from (11), (15), (16), depends on ε, since the constant K in (12) depends on ε.

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REFERENCES

1. T. D. Venttsel, *DAN*, **117**, No. (1957).

2. S. D. Eidelman, *Matem. sborn.*, **53**, No. 1 (1961).

3. T. D. Venttsel, Dissertation, Moscow, 1958.

Note: Figure translations are in progress. See original paper for figures.

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