B. S. Pavlov

1. Consider the differential operator

1961

SovietRxiv

Abstract

Full Text

B. S. Pavlov

ON A NON-SELF-ADJOINT OPERATOR $-y'' + q(x)y$ ON THE HALF-AXIS

(Presented by Academician V. I. Smirnov, 23 VI 1961)

1. Consider the differential operator

$$G = -y'' + g(x)y; \qquad y(0) - hy'(0) = 0, \tag{1}$$

in the Hilbert space $L_2(0, \infty)$, where the continuous function $q(x)$ and the number h are, in general, complex. It is known* that under the condition

$$\int_0^\infty x|g(x)|\,dx < \infty \tag{2}$$

the continuous spectrum of the operator (1) fills the half-axis $[0, \infty)$. The residual spectrum is absent. In addition, isolated eigenvalues of finite rank may enter the spectrum of the operator. The set of such eigenvalues is bounded and can have limit points only on the half-axis $[0, \infty)$. Positive eigenvalues are absent. Let us also note that in the case of a real potential $q(x)$ all eigenvalues λ_s satisfy the condition $\operatorname{Im} h \cdot \operatorname{Im} \lambda_s \leqslant 0$.

M. A. Naimark ([1]) showed that under the conditions

$$\max_{0 \le x < \infty} |g(x)| e^{\varepsilon x} < \infty; \qquad \int_0^\infty |g(x)| e^{\varepsilon x}\,dx < \infty, \quad \varepsilon > 0, \tag{3}$$

the discrete spectrum of the operator (1) consists of a finite number of eigenvalues. Moreover, the number of exceptional points (see the definition in Sec. 2) is also finite. As was shown in ([1],[4]), under condition (2) the expansion theorem in eigenfunctions of the operator (1) is valid if the total number of exceptional points is finite. At the same time it is known that in the case of real h and $q(x)$ the finiteness of the number of exceptional points is guaranteed by condition (2).

There was an opinion that the gap in the nature of the conditions for the self-adjoint and non-self-adjoint cases is not essential and can be eliminated. In addition, J. Schwartz (see ([5])) put forward the hypothesis that under very general conditions on the growth of the coefficients the spectrum of a singular differential operator consists of a finite or countable number of analytic arcs and no more than a countable set of eigenvalues accumulating at their ends. He also suggested that, when approaching normally an interior point λ_0 of each such arc, the resolvent R of the operator (1) has the estimate** $\|R_\lambda\| \leqslant C|\lambda - \lambda_0|^{-1}$. In the present note it is shown that, even for real $q(x)$ and complex h, the conditions (3) for finiteness of the number of exceptional points cannot be substantially weakened. Our results may also be regarded as a refutation of both parts of Schwartz' s hypothesis.

2. Consider the differential equation

$$-y'' + q(x)y = \lambda y \qquad (\lambda = k^2,\ \operatorname{Im} k \geqslant 0;\ 0 \leqslant x < \infty). \tag{4}$$

If condition (2) is fulfilled, then for every non-real k this equation has a unique solution $f(x,k)$ belonging to $L_2(0,\infty)$ and satisfying the condition $\lim_{x\to+\infty} e^{-ikx} f(x,k) = 1$. In this case $f(x,k)$ and $f'(x,k)$

* See, for example, ([1]-[4]). In these works the more restrictive condition

$$\int_0^\infty (1+x^2)|q(x)|\,dx < \infty,$$

was imposed, but, as is easily seen, everything stated is also valid under condition (2).

** It can be shown that this condition is equivalent to the fact that λ_0 is not an exceptional point.

regular in the upper half-plane $\operatorname{Im} k > 0$ and continuous in the closed upper half-plane $\operatorname{Im} k \geq 0$. The point $\lambda = k_0^2$ will be called a singular point of operator (1) if k_0 is a root of the function $D(k) \equiv f(0,k) - hf'(0,h)$. Let us note that our definition distinguishes points lying on different shores of the cut $[0,\infty)$. The multiplicity of a singular point will mean the multiplicity of the root $D(k)$.

Obviously, all singular points not lying on $[0,\infty)$ are eigenvalues of finite rank of operator (1). The singular points lying on the shores of the cut $[0,\infty)$ are not eigenvalues, but, as it turns out, play a special role in the expansion theorem (see (1)).

We shall need the following simple assertions:

Lemma 1. *Let $q(x)$ be a continuous complex function satisfying condition (2). Then the set of accumulation points of the eigenvalues of operator (1) is bounded, closed, and consists of singular points. The set of all real singular points is bounded, closed, and has linear measure zero.*

Lemma 2. *If* $\operatorname{Im} q(x) \equiv 0$, $\operatorname{Im} h \neq 0$, *then, under condition (2),* $\lambda = 0$ *is not a singular point of operator (1). The eigenvalues* λ_s *of the operator satisfy the condition*

$$\sum_{s=0}^{\infty}(-\operatorname{Im}\lambda_s \cdot \operatorname{Im} h) < \infty$$

(counting rank). The real singular points lie only on the upper shore of the cut if $\operatorname{Im} h < 0$, *and only on the lower one if* $\operatorname{Im} h > 0$.

Let us note a criterion for the finiteness of the number of singular points of operator (1):

Theorem 1. *Let* $q(x)$ *be a continuous complex function such that*

$$b_l = \int_0^{\infty} |q(x)|\, x^{l+1}\, dx < \infty, \qquad l = 0, 1, 2, \dots, \tag{5}$$

and, for some $c > 0$, $M > 0$, *the numbers* $m_l = c b_l (l+1)^{-1} M^{-l}$ *satisfy the condition*

$$\int_1^{\infty} r^{-2} \ln T(r)\, dr = \infty, \qquad \text{where } T(r) \equiv \sup_{l \geq 0} r^l m_l^{-1}. \tag{6}$$

*Then the number of singular points of operator (1) and their total multiplicity are finite.**

Let us note that the conditions of Theorem 1 guarantee the quasi-analyticity of the function $D(k)$ on the real axis, whereas condition (3) guarantees its analyticity in the half-plane $\operatorname{Im} k > -\varepsilon$.

3. The assertion of Theorem 1 may become false if condition (5) is retained but condition (6) is dropped. Namely:

Theorem 2. *For every number* $k_0^2 > 0$ *there exists a continuous real function* $q(x)$, *satisfying conditions (5), such that the corresponding operator of the form (1), for some non-real* h, *has an infinite sequence of eigenvalues accumulating at the point* k_0^2. *The point* k_0^2 *is a singular point of infinite multiplicity for operator (1).*

Let $\varphi_\alpha(x, \lambda)$ and $\psi_\alpha(x, \lambda)$ be solutions of the differential equation

$$-y'' + q(x)y = \lambda y, \qquad \operatorname{Im} q(x) \equiv 0, \tag{4'}$$

satisfying the initial conditions

$$\varphi_\alpha(0,\lambda) = \sin\alpha, \quad \varphi'_\alpha(0,\lambda) = -\cos\alpha, \quad \psi_\alpha(0,\lambda) = \cos\alpha, \quad \psi'_\alpha(0,\lambda) = \sin\alpha; \qquad 0 \leq \alpha < 2\pi.$$

Every other solution of equation $(4')$ can be represented as a linear combination of $\varphi_\alpha(x, \lambda)$ and $\psi_\alpha(x, \lambda)$. If $q(x)$ satisfies condition (2), then to each non-real λ there corresponds a unique, up to normalization, solution of equation $(4')$

belonging to $L_2(0,\infty)$. In view of what was said above, it is represented in the form

$$\psi_\alpha(x,\lambda)+m_\alpha(\lambda)\varphi_\alpha(x,\lambda).$$

Here $m_\alpha(\lambda)$ is a regular function in each of the half-planes $\operatorname{Im}\lambda>0$, $\operatorname{Im}\lambda<0$, having negative imaginary part in the upper half-plane and positive

* An analogous assertion is valid in the three-dimensional case.

in the lower half-plane, taking continuous boundary values on the cut $[0,\infty)$ and having a finite number of simple poles on the half-axis $(-\infty,0)$. The spectral measure of the operator (1) for $h=-\operatorname{tg}\alpha$ is determined by $m_\alpha(\lambda)$:

$$\rho_\alpha(\lambda)=\lim_{\delta\to 0,\ \delta>0}\int_0^\lambda\{-\operatorname{Im}m_\alpha(u+i\delta)\}\,du.$$

Conversely, $\{m_\alpha(\lambda)\}$ is uniquely determined by $\rho_\alpha(\lambda)$. For example,

$$m_\alpha(\lambda)=-\operatorname{ctg}\alpha+\frac{1}{\pi}\int_{-\infty}^{\infty}(\lambda-\xi)^{-1}\,d\rho_\alpha(\xi),$$

if $\alpha\neq 0$; $m_\alpha(\lambda)$ is called the Weyl function of the differential operator (1).

In the proof of Theorem 2 the following is used.

Lemma 3. *Let $m(k)$ be a function regular in the upper half-plane $\operatorname{Im}k>0$ and in neighborhoods of the points $k=0$ and $k=\infty$, satisfying the conditions: 1) $\operatorname{Im}m(k)=0$, when $ik\le 0$; $\operatorname{Im}m(k)=-\operatorname{Im}m(-k)<0$, when $k>0$; 2) $im'(0)>0$; $m(k)=-ik^{-1}+O(k^{-2})$ as $k\to\infty$; 3) $m(k)$ is continuous in the closed upper half-plane together with the first n derivatives ($n\le\infty$). Then there exists a differential operator of the form (1) with continuous potential $q(x)$, satisfying the conditions*

$$\operatorname{Im}q(x)\equiv 0,\qquad \int_0^\infty x^l|q(x)|\,dx<\infty,\qquad l=0,1,2,\dots,n-2,\qquad (5')$$

such that $m(\sqrt{\lambda})$ is the Weyl function of this operator for $\alpha=\pi/2$.

The existence of a differential operator with continuous potential is easily obtained from 1) and 2) using the theory of the inverse spectral problem ([8]). To verify fulfillment of the condition $(5')$ on the decrease of the potential, some results of V. A. Marchenko on the inverse problem of scattering theory ([6]) are used. If λ does not lie on the half-axis $[0,\infty)$, then $m_{\pi/2}(\lambda)=f(0,k)/f'(0,k)$, where $f(x,k)$ is the same solution as in item 2; $k=\sqrt{\lambda}$. Therefore the eigenvalues of the operator (1) coincide with the roots of the difference $m_{\pi/2}(\lambda)-h$ and have rank equal to the multiplicity of the corresponding root. Hence it is clear that the proof of Theorem 2 reduces to the construction of a function

$m(k)$ satisfying all the conditions of Lemma 3 (with $n = \infty$) and taking some complex value h at an infinite set of points accumulating at the point k_0^2. The Weyl function $m_{\pi/2}(\lambda)$ can be constructed so that the roots of the difference $m_{\pi/2}(\lambda) - h$ are simple and $h \neq \pm i$. We shall show that in this case also the series in eigenfunctions of the operator (1) diverges for an arbitrary finite function.

Let $u(x)$ and $v(x)$ be arbitrary finite functions; $\varphi_h(x, \lambda)$ is the solution of equation (4′), satisfying the initial conditions:

$$\varphi_h(0, \lambda) = -h(1 + h^2)^{-1/2}, \qquad \varphi_h'(0, \lambda) = (1 + h^2)^{-1/2}.$$

Further, let

$$m_h(\lambda) = (1 + m_{\pi/2}(\lambda)h)(h - m_{\pi/2}(\lambda))^{-1}, \qquad \tilde{u}(\lambda) = \int_0^\infty u(x)\varphi_h(x, \lambda)\, dx,$$

$$\tilde{v}(\lambda) = \int_0^\infty \overline{v(x)}\, \varphi_h(x, \lambda)\, dx.$$

The functions $\tilde{u}(\lambda)$ and $\tilde{v}(\lambda)$ are entire. Let γ be some contour enclosing the spectrum of the operator (1). Then (7) gives the "Parseval equality"

$$(u, v) = \int_0^\infty u(x)\overline{v(x)}\, dx = \frac{1}{2\pi} \oint_\gamma \tilde{u}\, \tilde{v}\, m_h(\lambda)\, d\lambda.$$

Contracting the contour to the real axis and extracting the residues at the poles, we obtain

$$(u, v) = \frac{1}{2\pi} \int_{\gamma_N} \tilde{u}(\lambda)\tilde{v}(\lambda) m_h(\lambda)\, d\lambda - \sum_1^N (1 + h^2)[m_{\pi/2}'(\lambda_i)]^{-1} \tilde{u}(\lambda_i)\tilde{v}(\lambda_i).$$

Here γ_N is a contour enclosing the continuous spectrum and all eigenvalues beginning with λ_{N+1}. The second term is a partial sum of the series in eigenfunctions. In our case there is no finite limit of the partial sums, since the general term of the series tends to infinity.

Indeed, $\tilde{u}(\lambda)\tilde{v}(\lambda)$ can have at the point $\lambda = k_0^2$ a zero of no more than finite order, while $m_{\pi/2}(\lambda) - h \to 0$ as $\lambda \to k_0^2$ faster than any power, since $\lambda = k_0^2$ is a singular point of infinite order of the operator (1).

4. By the same method one proves:

Theorem 3. *There exists a differential operator of the form (1), with a continuous real potential $q(x)$ satisfying the conditions (5), which, for some complex h, has an infinite set of eigenvalues accumulating to a certain perfect set of linear measure zero and of the cardinality of the continuum, situated on the positive half-axis.*

Let us note that the corresponding construction makes it possible, under the conditions of Theorem 3, to obtain eigenvalues of arbitrarily high rank.

5. If condition (2) is abandoned, then the assertion of Lemma 1 becomes, generally speaking, false.

Let the potential $q(x)$ be such that the operator (1) is semibounded for real h, and the continuous spectrum fills the half-axis $[0, \infty)$. The Weyl function of such an operator $m_{\pi/2}(\lambda)$ is regular in the λ-plane cut along $[0, \infty)$, with the exception of at most a finite number of poles on the half-axis $(-\infty, 0)$, and satisfies the condition

$$\operatorname{Im} m_{\pi/2}(\lambda) \times \operatorname{Im} h \leq 0.$$

If $m_{\pi/2}(\lambda)$ is bounded for all λ belonging to the half-plane $\operatorname{Re} \lambda > 0$, then $m_{\pi/2}(\lambda)$ has definite angular boundary values almost everywhere on the edges of the cut $[0, \infty)$; moreover the set $\{\lambda_0\}$, where the angular boundary values are equal to h, has linear measure zero [(9)].

We shall call a point $\lambda_0 > 0$, lying on the upper edge of the cut $[0, \infty)$, a singular point if $m_{\pi/2} - h \to 0$ when $\lambda \to \lambda_0$ in some angle with vertex at λ_0, situated entirely in the upper half-plane. If the point λ_0 lies on the lower edge of the cut, one should take an angle situated entirely in the lower half-plane.

If condition (2) is fulfilled, then the new definition of a singular point introduced by us coincides with the former one (see § 2).

Obviously, if $m_{\pi/2}(\lambda)$ is bounded in the half-plane $\operatorname{Re} \lambda > 0$, then the singular points still form a set of linear measure zero on the half-axis. But it can no longer be said of this set that it consists of accumulation points of eigenvalues.

Theorem 4. *There exists a differential operator of the form (1), semibounded for real h, with a continuous real potential such that, for some complex h,* $\operatorname{Im} h < 0$, *the continuous spectrum fills the positive half-axis, the eigenvalues form a bounded set whose limit points fill an interval lying on the upper edge of the cut* $[0, \infty)$*, while the singular points lying on the upper edge of the cut* $[0, \infty)$ *form a bounded set of linear measure zero.*

The author expresses deep gratitude to M. Sh. Birman for posing the problem and for constant attention to the work, and also to V. P. Khavin for useful discussions of the results.

Leningrad State University
named after A. A. Zhdanov

Received
1 VI 1961

CITED LITERATURE

[1] M. A. Naimark, *Tr. Mosk. matem. obshch.*, **3**, 181 (1954).
[2] I. M. Gel' fand, *UMN*, **7**, no. 6 (1952).
[3] R. M. Martirosyan, *Izv. AN ArmSSR*, **10**, no. 1, 85 (1957).
[4] B. Ya. Levin, *DAN*, **106**, 187 (1957).

[5] N. Danford, *Sborn. per. Matematika*, no. 1, 1960.
[6] V. A. Marchenko, *Tr. Mosk. matem. obshch.*, **1**, 327 (1952).
[7] V. A. Marchenko, *Matem. sborn.*, **52**, no. 2 (1960).
[8] I. M. Gel' fand and B. M. Levitan, *Izv. AN SSSR, ser. matem.*, **15**, 309 (1951).
[9] I. I. Privalov, *Boundary Properties of Analytic Functions*, L., 1950.

Note: Figure translations are in progress. See original paper for figures.