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Abstract

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L. A. NAZAROVA

INTEGER REPRESENTATIONS OF THE FOUR-GROUP

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By an integer representation of a group G one means, as usual, a homomorphism of the group G into the group of automorphisms of some free abelian group (lattice) R, or, what is the same thing, into the group of integral unimodular matrices. At present the representations of cyclic groups of prime order ([1,2]) and of order four ([3]) have been completely described; moreover, for these groups there exist only finitely many indecomposable representations. In particular, for the cyclic group of order two there are three indecomposable representations: $(+1)$, (-1), $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and the decomposition into them is unique up to the arrangement of the boxes along the main diagonal. It is known, on the other hand, that if the group G contains a noncyclic Sylow subgroup, then there are infinitely many indecomposable representations of the group G ([4,5]); however, for no group of this kind has a description of all indecomposable representations been given.

In the present paper all indecomposable representations of the group given by the relations $a^2 = b^2 = e,\ ab = ba$ are described.

To specify a representation of this group means to specify integral unimodular matrices A and B such that $A^2 = B^2 = E,\ AB = BA$.

In the lattice R we distinguish the sublattice S—the kernel of the operator $A - B$ —and extend its basis to a basis of the whole lattice. Then the matrices A and B will have the form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} A_{11} & B_{12} \\ 0 & -A_{22} \end{pmatrix}. \tag{1}$$

Using the result on representations of the cyclic group of order two, we decompose A_{11} and A_{22} into indecomposable boxes. Consider the case when A_{11} and A_{22} split only into the boxes $(+1)$ and (-1). The matrices A and B have the form

$$A = \begin{pmatrix} E & 0 & 0 & A_{14} \\ 0 & -E & A_{23} & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & -E \end{pmatrix}, \qquad B = \begin{pmatrix} E & 0 & -B_{13} & 0 \\ 0 & -E & 0 & B_{24} \\ 0 & 0 & -E & 0 \\ 0 & 0 & 0 & E \end{pmatrix}. \tag{2}$$

Let us introduce the matrix D: replace in $A_{14}, A_{23}, B_{13}, B_{24}$ all even numbers by zeros and all odd numbers by ones, and then construct the matrix

$$D = \begin{pmatrix} D_1 & D_4 \\ D_2 & D_3 \end{pmatrix}, \tag{3}$$

where D_1 is the matrix obtained from B_{13}, D_2 from A_{23}, D_3 from B_{24}, D_4 from A_{14} in the indicated way. We shall regard D as a matrix over the field of two elements.

It turns out that if two representations A, B and A', B' have equal $A_{11} = A'_{11}$, $A_{22} = A'_{22}$, $D = D'$, then they are equivalent and the transformation the similarity is effected by a matrix of the form

$$C = \begin{pmatrix} E & C_{12} \\ 0 & E \end{pmatrix},$$

and, conversely, if two representations are equivalent with a transforming matrix of the indicated form, then they have equal D.

If the matrices of the representation A, B are equivalent to A', B' with transforming matrix

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}, \tag{4}$$

then the matrix D is transformed by means of two matrices $\overline{C}_1$ and $\overline{C}_2$ as follows: $\overline{C}_1^{-1} D \overline{C}_2$, where $\overline{C}_1$ and $\overline{C}_2$ are obtained from the matrices C_1 and C_2 by replacing even numbers by zeros and odd numbers by ones; and conversely, if $D' = \overline{C}_1^{-1} D \overline{C}_2$, then there exist such matrices A' and B' and such matrices C_1 and C_2 that A, B are equivalent to A', B', the equivalence is effected by a matrix of the form (4), and D' is obtained from A' and B' in the indicated manner.

Denote $n_i = \max(a_i, b_i)$, where a_i is the number of rows of D_i, b_i is the number of columns of D_i $(i = 1, 2, 3, 4)$; $d = \sum_{i=1}^{4} r_i$, where $r_i = 2R_i - \min(a_i, b_i)$, and R_i is the rank of D_i $(i = 1, 2, 3, 4)$. It turns out that, for indecomposable representations, $d = n - 2$ and $d = n$, where n is the dimension of the representation.

Let $d = n-2,\ r_1 \neq n_1$. Then the following cases are possible: 1) $r_1 = n-2$; 2) $r_1 = n_1-1,\ r_2 = n_2-1$; 3) $r_1 = n_1-1,\ r_3 = n_3-1$; 4) $r_1 = n_1-1,\ r_4 = n_4-1$.

In case 1), corresponding to representation dimension $n \equiv 0 \pmod 4$, D is reduced to the form

$$D_1 = \begin{pmatrix} 010\ldots0 \\ 001\ldots0 \\ \cdot\ \cdot\ \cdot\ \cdot \\ 000\ldots1 \\ 000\ldots0 \end{pmatrix}, \qquad D_2 = D_3 = D_4 = E.$$

In case 2), for dimensions $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, we obtain one indecomposable representation for each. D is reduced to the form

$$D_1 = \begin{pmatrix} 10\ldots00 \\ 01\ldots00 \\ \cdot\ \cdot\ \cdot\ \cdot \\ 00\ldots10 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 010\ldots0 \\ 001\ldots0 \\ \cdot\ \cdot\ \cdot\ \cdot \\ 000\ldots1 \end{pmatrix}; \qquad D_1^* = \begin{pmatrix} 00\ldots00 \\ 10\ldots00 \\ \cdot\ \cdot\ \cdot\ \cdot \\ 00\ldots01 \end{pmatrix}, \qquad D_2^* = \begin{pmatrix} 10\ldots00 \\ 01\ldots00 \\ \cdot\ \cdot\ \cdot\ \cdot \\ 00\ldots10 \end{pmatrix},$$

$$D_3 = E, \qquad D_4 = E; \qquad D_3^* = E, \qquad D_4^* = E.$$

In case 3), for representations of dimension $n \equiv 2 \pmod 4$, we obtain two more indecomposable representations

$$D_1 = \begin{pmatrix} 00\ldots00 \\ 10\ldots00 \\ \cdot\ \cdot\ \cdot\ \cdot \\ 00\ldots01 \end{pmatrix}, \qquad D_3 = \begin{pmatrix} 10\ldots00 \\ 01\ldots00 \\ \cdot\ \cdot\ \cdot\ \cdot \\ 00\ldots10 \end{pmatrix}; \qquad D_1^* = \begin{pmatrix} 10\ldots00 \\ 01\ldots00 \\ \cdot\ \cdot\ \cdot\ \cdot \\ 00\ldots10 \end{pmatrix}, \qquad D_3^* = \begin{pmatrix} 00\ldots00 \\ 10\ldots00 \\ \cdot\ \cdot\ \cdot\ \cdot \\ 00\ldots01 \end{pmatrix},$$

$$D_2 = E, \qquad D_4 = E; \qquad D_2^* = E, \qquad D_4^* = E.$$

In case 4), for representations of dimensions $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, we obtain one indecomposable representation for each:

$$D_1 = \begin{pmatrix} 00\ldots00 \\ 10\ldots00 \\ \cdot\ \cdot\ \cdot\ \cdot \\ 00\ldots10 \\ 00\ldots01 \end{pmatrix}, \qquad D_4 = \begin{pmatrix} 10\ldots00 \\ 01\ldots00 \\ \cdot\ \cdot\ \cdot\ \cdot \\ 00\ldots01 \\ 00\ldots00 \end{pmatrix}; \qquad D_1^* = \begin{pmatrix} 10\ldots00 \\ 01\ldots00 \\ \cdot\ \cdot\ \cdot\ \cdot \\ 00\ldots10 \end{pmatrix}, \qquad D_4^* = \begin{pmatrix} 010\ldots00 \\ 001\ldots00 \\ \cdot\ \cdot\ \cdot\ \cdot \\ 000\ldots01 \end{pmatrix},$$

$$D_2 = E, \qquad D_3 = E; \qquad D_2^* = E, \qquad D_3^* = E.$$

Let $d = n - 2,\ r_2 \neq n_2$. Analogously to the preceding case, we obtain one more irreducible representation of dimensions $n \equiv 0 \pmod 4$, $n \equiv 1 \pmod 4$, $n \equiv 3 \pmod 4$, and two representations of dimension $n \equiv 2 \pmod 4$.

Let $d = n - 2,\ r_3 \neq n_3$. We obtain one more irreducible representation of dimensions $n \equiv 0 \pmod 4$, $n \equiv 1 \pmod 4$, $n \equiv 3 \pmod 4$.

And, finally, the case $n = d - 2,\ r_4 \neq n_4$ adds the unique irreducible representation of dimension $n \equiv 0 \pmod 4$.

Let $n = d$. This case, as is easy to see, can occur only for dimension $n \equiv 0 \pmod 4$. D is reduced to the form:

$$D = \begin{pmatrix} E & D_4 \\ E & E \end{pmatrix},$$

where, if $A,\ B$ are equivalent to $A',\ B'$, then D_4 is equivalent to D'_4, and conversely, if D_4 is equivalent to D'_4, then there exist matrices $A',\ B'$ equivalent to $A,\ B$ and such that, under the indicated construction, the matrix D'_4 corresponds to them.

The representation is irreducible if D_4 is irreducible. The matrix D_4, as a matrix over the field of two elements, can be reduced to Frobenius normal form. D_4 is irreducible if its characteristic polynomial is irreducible or a power of an irreducible one. Hence there exist

$$\frac{1}{4} \sum_{d \mid n_4} \varphi(d) 2^{n_4/d} - 1$$

(where φ is Euler' s function) irreducible representations of this type.

Let us now consider the most general case. If the representation is irreducible, then three values of d are possible: $n-2,\ n,\ n-4$, and A and B can be reduced to the form:

$$A = \left(\begin{array}{ccc|ccc} E & 0 & 0 & 0 & A_{15} & A_{16} \\ 0 & -E & 0 & A_{24} & 0 & A_{26} \\ 0 & 0 & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & A_{34} & A_{35} & 0 \\ \hline & & & E & 0 & 0 \\ & 0 & & 0 & -E & 0 \\ & & & 0 & 0 & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \end{array}\right), \qquad B = \left(\begin{array}{ccc|ccc} E & 0 & 0 & B_{14} & 0 & B_{16} \\ 0 & -E & 0 & 0 & B_{25} & B_{26} \\ 0 & 0 & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & B_{34} & B_{35} & 0 \\ \hline & & & -E & 0 & 0 \\ & 0 & & 0 & E & 0 \\ & & & 1 & 0 & \begin{smallmatrix} 0 & -1 \\ -1 & 1 \end{smallmatrix} \end{array}\right).$$

If $d = n - 2,\ r_1 = n_1 - 1,\ r_2 = n_2 - 1$, then we obtain one irreducible representation of dimension $n \equiv 1 \pmod 4$ of the following form: A_{11} decomposes into boxes of all three kinds, A_{22} only into boxes $(+1)$ and (-1),

$$A_{24} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \qquad B_{14} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \qquad A_{34} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

$$A_{15} = E, \qquad B_{25} = E, \qquad B_{34} = 0, \qquad A_{35} = B_{35} = 0,$$

and analogously—one irreducible representation of dimension $n \equiv 3 \pmod 4$, where A_{11} decomposes into boxes $(+1)$ and (-1), A_{22} into boxes of three kinds. If $d = n - 2,\ r_1 = n_1 - 1,\ r_4 = n_4 - 1$, then analogously we obtain one more irreducible representation of dimension $n \equiv 1 \pmod 4$, where A_{11} decomposes into boxes $(+1)$, (-1), A_{22} into boxes of three kinds, and one of dimension $n \equiv 3 \pmod 4$, where A_{11} decomposes into boxes of three kinds, A_{22} into $(+1)$ and (-1).

Passing to the next cases, we obtain two more irreducible representations each of dimensions $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$.

Let $d = n - 4$. In this case we obtain two irreducible representations of dimension $n \equiv 4 \pmod 8$. The first has the form: A_{11} decomposes into boxes $(+1)$ and (-1), A_{22} into boxes of all three kinds,

$$A_{15} = \begin{pmatrix} A'_{15} & 0 \\ 0 & \begin{smallmatrix} 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{smallmatrix} \end{pmatrix}, \qquad A_{24} = \begin{pmatrix} \begin{smallmatrix} 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{smallmatrix} & 0 \\ 0 & A'_{24} \end{pmatrix}, \qquad A_{16} = \begin{pmatrix} 0 & 0 \\ \dots & \dots \\ 0 & 0 \\ 1 & 1 \end{pmatrix}, \qquad B_{26} = \begin{pmatrix} 0 & 0 \\ \dots & \dots \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ \dots & \dots \\ 0 & 0 \end{pmatrix},$$

$$B_{14} = \begin{pmatrix} \begin{smallmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{smallmatrix} & 0 \\ 0 & B'_{14} \end{pmatrix}, \qquad B_{25} = \begin{pmatrix} B'_{25} & 0 \\ 0 & \begin{smallmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 1 \end{smallmatrix} \end{pmatrix},$$

$$A'_{15} = A'_{24} = B'_{14} = B'_{25} = E, \qquad A_{26} = 0, \quad B_{16} = 1.$$

The second is analogous to the first, where A_{11} decomposes into boxes of all three types, and A_{22} into boxes $(+1)$, (-1).

Analogously we obtain two indecomposable representations of dimension $n \equiv 0 \pmod 8$.

Let now $n = d$. Then there is singled out the case when the characteristic polynomial D_4 decomposes into linear factors and the matrix D_4 is brought to Jordan normal form. Obviously, there exists only one such matrix. In this case, for dimension $n \equiv 2 \pmod 4$ we have two indecomposable representations. The first has the form: A_{11} decomposes into boxes of all three types, and A_{22} into boxes $(+1)$ and (-1),

$$B_{14} = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \qquad B_{34} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix}, \qquad A_{15} = A_{24} = B_{25} = E,$$

$$B_{35} = A_{34} = A_{35} = 0.$$

The second is analogous to the first, where A_{11} decomposes into boxes $(+1)$ and (-1), and A_{22} into boxes of all three types. This same case gives one more representation of dimension $n \equiv 0 \pmod 4$: A_{11} and A_{22} decompose into boxes of all three types,

$$B_{14} = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}, \qquad B_{16} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ \dots & \dots \\ 0 & 0 \end{pmatrix}, \qquad B_{34} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{pmatrix},$$

$$B_{25} = A_{15} = A_{24} = E, \qquad A_{16} = A_{26} = 0,$$

$$A_{34} = A_{35} = 0, \qquad B_{26} = 0, \quad B_{35} = 0.$$

Summarizing all the results obtained, we have: for odd dimensions, starting with 5, there are 8 indecomposable representations of each dimension; for $n \equiv 2 \pmod 4$, there are 6; for dimension $n \equiv 0 \pmod 4$, there are

$$6 + \sum_{d/m} \varphi(d)\, 2^{m/d}$$

(where φ is Euler' s function, and the dimension is $m - \frac{1}{4}$) indecomposable representations of each dimension of this form.

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Leningrad State University
named after A. A. Zhdanov

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Note: Figure translations are in progress. See original paper for figures.