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Abstract

Full Text

MATHEMATICS

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ON A PRIORI ESTIMATES FOR SOME BOUNDARY-VALUE PROBLEMS

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No. 1. Let Ω be a rectangular parallelepiped in the space $E_n : a_i < x_i < b_i$ (the case of infinite a_i and b_i is not excluded). In (1) the concept of the spaces $\mathfrak{L}_p^{l_1 \dots l_n}(\Omega)$ and $\mathfrak{W}_p^{l_1 \dots l_n}(\Omega)$ was introduced and their properties were investigated. We shall now introduce the spaces $L_p^{l_1 \dots l_n}(\Omega)$ and $W_p^{l_1 \dots l_n}(\Omega)$. Let

$$D_i^k = \partial^k / \partial x_i^k, \quad \lambda_i = l_i - [l_i]$$

and

$$\|f\|_{L_p^{l_i}(\Omega)}^p = \begin{cases} \int_{\Omega} dx \int_0^{b_i - x_i} |D_i^{[l_i]} f(x_1 \dots x_i + h \dots x_n) - D_i^{[l_i]} f(x)|^p \frac{dh}{h^{1+p\lambda_i}}, & \text{for } \lambda_i > 0, \\ \int_{\Omega} |D_i^{l_i} f(x)|^p dx, & \text{for } \lambda_i = 0. \end{cases}$$

We define the spaces $L_p^{l_1 \dots l_n}(\Omega)$ and $W_p^{l_1 \dots l_n}(\Omega)$ as the closure of the set of smooth functions equal to zero for large $|x|$, in the norms

$$\|f\|_{L_p^{l_1 \dots l_n}(\Omega)} = \sum_{i=1}^n \|f\|_{L_p^{l_i}(\Omega)}, \quad \|f\|_{W_p^{l_1 \dots l_n}(\Omega)} = \|f\|_{L_p(\Omega)} + \|f\|_{L_p^{l_1 \dots l_n}(\Omega)}.$$

For nonintegral l_i these spaces coincide with $\mathfrak{L}_p^{l_1 \dots l_n}(\Omega)$ and $\mathfrak{W}_p^{l_1 \dots l_n}(\Omega)$.

Theorem 1. Let $f \in W_p^{l_1 \dots l_n}(\Omega)$, $0 < s < n$, and suppose that there exist integers ν_i such that

$$\mu = 1 - \sum_{i=1}^n \frac{\nu_i}{l_i} - \frac{1}{p} \sum_{i=s+1}^n \frac{1}{l_i} > 0.$$

Then

$$D_1^{\nu_1} \dots D_n^{\nu_n} f \in \mathfrak{W}_p^{l_1 \mu \dots l_s \mu}(\Omega_s), \quad \|D_1^{\nu_1} \dots D_n^{\nu_n} f\|_{\mathfrak{W}_p^{l_1 \mu \dots l_s \mu}(\Omega_s)} \leq C_1 \|f\|_{W_p^{l_1 \dots l_n}(\Omega)},$$

where Ω_s is the section of Ω by the hyperplane $x_{s+1} = \text{const}$, $x_n = \text{const}$.

This theorem is a generalization of the well-known results of Aronszajn, Babich, Slobodetskii, and Gagliardo on the boundary properties of functions from W_p^l for integral l . It admits a converse.

Theorem 2. Let $\varphi(x)$ be an infinitely differentiable function of one variable, defined on the positive half-axis and vanishing together with all its derivatives at $x = 0$, and also for sufficiently large x , and let $\varphi^*(x)$ be its extension to the whole axis, with $\varphi^*(x) = 0$ for $x < 0$. Then, for all $l \geq 0$, except $l = k + 1/p$, where $k \geq 0$ is an integer, the inequality

$$\|\varphi^*\|_{\mathfrak{L}_p^l(E_1)} \leq C_2 \|\varphi\|_{\mathfrak{L}_p^l(0,\infty)}$$

holds.

For $l = k + 1/p$ this inequality is false.

No. 2. Consider, in the half-space $x_n \geq 0$, the boundary-value problems:

$$\Delta u = f(x), \quad u|_{x_n=0} = \varphi(x_1 \dots x_{n-1}), \quad (1)$$

$$\Delta v = f(x), \quad \left. \frac{\partial v}{\partial x_n} \right|_{x_n=0} = \psi(x_1 \dots x_{n-1});$$

$$u_t - \Delta u = f(x, t), \quad u|_{t=0} = u_0(x), \quad u|_{x_n=0} = \varphi(x_1 \dots x_{n-1}, t), \quad (2)$$

$$v_t - \Delta v = f(x, t), \quad v|_{t=0} = v_0(x), \quad \left. \frac{\partial v}{\partial x_n} \right|_{x_n=0} = \psi(x_1 \dots x_{n-1}, t),$$

and in the half-space $x_3 \geq 0$ of three-dimensional space E_3 the problem

$$\Delta \mathbf{u} = \text{grad } p + \mathbf{f}, \quad \text{div } \mathbf{u} = \rho(x), \quad \mathbf{u}|_{x_3=0} = \mathbf{a}(x_1, x_2). \quad (3)$$

Introduce the following notation: D_n is the half-space $x_n \geq 0$ of the space E_n ; \tilde{D}_n is the half-space $t \geq 0$ of the space E_n of the variables x_1, \dots, x_{n-1}, t ; A_{n+1} is the domain $x_n \geq 0, t \geq 0$ of the space E_{n+1} . Thus, the function $f(x)$ is given in D_n , $\varphi(x_1 \dots x_{n-1})$ and $\psi(x_1 \dots x_{n-1})$ in E_{n-1} , $f(x, t)$ in A_{n+1} , u_0 and v_0 in D_n , $\varphi(x_1 \dots x_{n-1}, t)$ and $\psi(x_1 \dots x_{n-1}, t)$ in \tilde{D}_n , etc. It is well known that, by extending in the proper way the functions $f(x)$, $f(x, t)$, $u_0(x)$, $v_0(x)$, and others to the whole space E_n or E_{n+1} , one can write the solutions of problems (1) and (2) explicitly in terms of electrostatic or heat potentials.

As for problem (3), it is a generalization of the first boundary-value problem for the stationary linear Navier–Stokes system and is easily reduced to the latter by introducing the new unknown vector

$$\mathbf{v} = \mathbf{u} - \text{grad } q(x),$$

where $q(x)$ is the solution of the problem

$$\Delta q(x) = \rho, \quad \left. \frac{\partial q}{\partial x_3} \right|_{x_3=0} = a_3(x_1, x_2).$$

Since \mathbf{v} can be expressed in terms of Odqvist hydrodynamic potentials (2), the vector \mathbf{u} is expressed as a sum of hydrodynamic and electrostatic potentials.

All these potentials can be estimated in the norms \mathfrak{L}_p^l and L_p^l (the heat potentials in the norms $\mathfrak{L}_{px,t}^{2l,l}$ and $L_{px,t}^{2l,l}$)* in terms of the corresponding norms of their densities. With the help of these estimates (which we do not write out), as well as Theorems 1 and 2 and some results of the work (1), the following propositions are proved:

Theorem 3. For the solutions of problems (1) the following inequalities hold:

$$\|u\|_{\mathfrak{L}_p^{l+2}(D_n)} \leq C_3(\|f\|_{\mathfrak{L}_p^l(D_n)} + \|\varphi\|_{\mathfrak{L}_p^{l+2-1/p}(E_{n-1})}),$$

$$\|u\|_{L_p^{l+2}(D_n)} \leq C_4(\|f\|_{L_p^l(D_n)} + \|\varphi\|_{\mathfrak{L}_p^{l+2-1/p}(E_{n-1})}),$$

$$\|v\|_{\mathfrak{L}_p^{l+2}(D_n)} \leq C_5(\|f\|_{\mathfrak{L}_p^l(D_n)} + \|\varphi\|_{\mathfrak{L}_p^{l+1-1/p}(E_{n-1})}),$$

$$\|v\|_{L_p^{l+2}(D_n)} \leq C_6(\|f\|_{L_p^l(D_n)} + \|\psi\|_{\mathfrak{L}_p^{l+1-1/p}(E_{n-1})}).$$

Theorem 4. For the solution of problem (3) the following inequalities hold:

$$\sum_{i=1}^3 \|u_i\|_{\mathfrak{L}_p^{l+2}(D_3)} + \|\rho\|_{\mathfrak{L}_p^{l+1}(D_3)} \leq$$

* By $\mathfrak{L}_{px,t}^{2l,l}$ one should understand the space $\mathfrak{L}_p^{2l\dots 2l,l}$, regarding t as the last coordinate, and by \mathfrak{L}_p^l , $\mathfrak{L}_p^{l,\dots,l}$.

$$\leq C_7 \left(\sum_{i=1}^3 \|f_i\|_{L_p^l(D_3)} + \|\rho\|_{L_p^{l+1}(D_3)} + \sum_{i=1}^3 \|a_i\|_{\mathfrak{L}_p^{l+2-1/p}(E_2)} \right),$$

$$\sum_{i=1}^3 \|u_i\|_{L_p^{l+2}(D_3)} + \|\rho\|_{L_p^{l+1}(D_3)} \leq$$

$$\leq C_8 \left(\sum_{i=1}^3 \|f_i\|_{L_p^l(D_3)} + \|\rho\|_{L_p^{l+1}(D_3)} + \sum_{i=1}^3 \|a_i\|_{\mathfrak{L}_p^{l+2-1/p}(E_2)} \right).$$

Theorem 5. For the solutions of problem (2) the following inequalities hold

$$\|u\|_{\mathfrak{L}_{p\alpha,t}^{2l+2,l+1}(A_{n+1})} \leq C_9 \left(\|f\|_{\mathfrak{L}_{p\alpha,t}^{2l,l}(A_{n+1})} + \|u_0\|_{\mathfrak{L}_p^{2l+2-2/p}(D_n)} + \|\varphi\|_{L_{p\alpha,t}^{2l+2-1/p,l+1-1/2p}(\bar{D}_n)} \right),$$

$$\|u\|_{L_{p\alpha,t}^{2l+2,l+1}(A_{n+1})} \leq C_{10} \left(\|f\|_{L_{p\alpha,t}^{2l,l}(A_{n+1})} + \|u_0\|_{\mathfrak{L}_p^{2l+2-2/p}(D_n)} + \|\varphi\|_{\mathfrak{L}_{p\alpha,t}^{2l+2-1/p,l+1-1/2p}(\bar{D}_n)} \right),$$

provided only that $l + 1 - 1/2p \neq m + 1/p$, where $m \geq 0$ is an integer, and

$$\|v\|_{\mathfrak{L}_{p\alpha,t}^{2l+2,l+1}(A_{n+1})} \leq C_{11} \left(\|f\|_{\mathfrak{L}_{p\alpha,t}^{2l,l}(A_{n+1})} + \|v_0\|_{\mathfrak{L}_p^{2l+2-2/p}(D_n)} + \|\psi\|_{\mathfrak{L}_{p\alpha,t}^{2l+1-1/p,l+1/2-1/2p}(\bar{D}_n)} \right),$$

$$\|v\|_{L_{p\alpha,t}^{2l+2,l+1}(A_{n+1})} \leq C_{12} \left(\|f\|_{L_{p\alpha,t}^{2l,l}(A_{n+1})} + \|v_0\|_{\mathfrak{L}_p^{2l+2-2/p}(D_n)} + \|\psi\|_{\mathfrak{L}_{p\alpha,t}^{2l+1-1/p,l+1/2-1/2p}(\bar{D}_n)} \right),$$

provided only that $l + 1/2 - 1/2p \neq m + 1/p$. If $l + 1 - 1/2p = m + 1/p$ or $l + 1/2 - 1/2p = m + 1/p$, the corresponding inequalities do not hold.

The indicated exceptional cases disappear if, instead of (2), one considers problems without initial data for $-\infty < t < \infty$.

No. 3. Let Ω be a bounded domain with sufficiently smooth boundary. Introduce the following notation:

$$N_l(f, \Omega) = \left(\int_{\Omega} dx \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+pl}} dy \right)^{1/p} \quad \text{for } 0 < l < 1;$$

$$N_1(f, \Omega) = \left(\int_{\Omega} dx \int_{A(x)} \frac{|f(x) - 2f(\frac{x+y}{2}) + f(y)|^p}{|x - y|^{n+p}} dy \right)^{1/p},$$

where $A(x)$ is the set of those $y \in \Omega$ such that $\frac{x+y}{2} \in \Omega$, and

$$N_l(f, \Omega) = \sum_{\sum \nu_i = \bar{l}} N_{\lambda} (D_1^{\nu_1} \dots D_n^{\nu_n} f(x), \Omega) \quad \text{for } l = \bar{l} + \lambda,$$

where $\bar{l} \geq 0$ is an integer and $0 < \lambda \leq 1$.

Now one can define the spaces $\mathfrak{W}_p^l(\Omega)$ and $\widetilde{W}_p^l(\Omega)$ as the closures of the set of smooth functions in the norms

$$\|f\|_{\mathfrak{W}_p^l(\Omega)} = \|f\|_{L_p(\Omega)} + N_l(f, \Omega),$$

$$\|f\|_{\widetilde{W}_p^l(\Omega)} = \begin{cases} \|f\|_{\mathfrak{W}_p^l(\Omega)}, & \text{for nonintegral } l, \\ \|f\|_{L_p(\Omega)} + \sum_{\sum \nu_i = \bar{l}} \|D_1^{\nu_1} \dots D_n^{\nu_n} f\|_{L_p(\Omega)}, & \text{for integral } l. \end{cases}$$

Theorem 6. If Ω is a rectangular parallelepiped, then the norms $\|f\|_{\mathfrak{W}_p^l(\Omega)}$ and $\|f\|_{\widetilde{W}_p^l(\Omega)}$ are equivalent to the norms $\|f\|_{\mathfrak{W}_p^{l..l}(\Omega)}$ and $\|f\|_{W_p^{l..l}(\Omega)}$, introduced above.

Theorem 7. Every function in $\widetilde{\mathfrak{W}}_p^l(\Omega)$ or $\widetilde{W}_p^l(\Omega)$ can be extended to the whole space E_n with preservation of equivalence of norms.

For the spaces $W_p^l(\Omega)$ this theorem has been proved (see (3,4)), so that what is essentially new here is the case when $f \in \widetilde{\mathfrak{W}}_p^l(\Omega)$ with integral l .

In the usual way (by means of parametrization) the spaces \mathfrak{W}_p^l and \widetilde{W}_p^l are defined if the functions are given on some smooth manifold, for example the boundary S of a domain. The preceding theorems generalize to the spaces of functions $\mathfrak{W}_{p(x_1)\dots(x_m)}^{l_1\dots l_m}(Q)$ and $\widetilde{W}_{p(x_1)\dots(x_m)}^{l_1\dots l_m}(Q)$, whose elements are given in the cylinder $Q = \Omega_1 \times \dots \times \Omega_m$, where the Ω_i are smooth manifolds and $x_i \in \Omega_i$. For example, if the Ω_i are domains, then $\mathfrak{W}_{p(x_1)\dots(x_m)}^{l_1\dots l_m}(Q)$ is the closure of the set of smooth functions in the norm

$$\|f\|_{\mathfrak{W}_{p(x_1)\dots(x_m)}^{l_1\dots l_m}(Q)}^p = \sum_{i=1}^m \int_{\Omega_1} dx_1 \dots \int_{\Omega_{i-1}} dx_{i-1} \int_{\Omega_{i+1}} dx_{i+1} \dots \int_{\Omega_m} \|f\|_{\mathfrak{W}_p^{l_i}(\Omega_i)}^p dx_m.$$

No. 4. Consider in a bounded n -dimensional domain Ω the following boundary-value problems:

$$\begin{aligned} \mathcal{L}(x)u &= f, & u|_S &= \varphi, \\ \mathcal{L}(x)v &= f, & \frac{\partial v}{\partial N} \Big|_S &= \psi; \end{aligned} \tag{4}$$

$$u_t - \mathcal{L}(x, t)u = f, \quad u|_{t=0} = u_0, \quad u|_S = \varphi,$$

$$v_t - \mathcal{L}(x, t)v = f, \quad v|_{t=0} = v_0, \quad \frac{\partial v}{\partial N} \Big|_S = \psi, \quad 0 \leq t \leq T, \quad (5)$$

where $\mathcal{L}(x)$ and $\mathcal{L}(x, t)$ are second-order elliptic operators with coefficients depending respectively on x and on (x, t) . Also consider, for $n = 3$, the problem

$$\Delta \mathbf{u} = \text{grad } p + \mathbf{f}, \quad \text{div } \mathbf{u} = \rho, \quad \mathbf{u}|_S = \mathbf{a}(x), \quad (6)$$

assuming that the necessary condition is satisfied

$$\int_{\Omega} \rho \, dx = \int_S a_n \, dS.$$

Following the arguments of Schauder ⁽⁵⁾, one can, with the aid of Theorems 3-5, obtain estimates for solutions of problems (4)-(6). Their difference from the inequalities of Theorems 3-5 consists in the fact that instead of the norms L_p^l and L_p^l , the norms \mathfrak{W}_p^l and \widetilde{W}_p^l occur everywhere. In addition, the right-hand side of the estimates for the solutions of problem (4) will contain the norms of these solutions in the space $L_p(\Omega)$.

It is necessary to note that some of the results presented above are known. Estimates for elliptic equations in the norms \widetilde{W}_p^l for integral l and in \widetilde{W}_2^l for arbitrary l have been obtained by a number of authors ⁽⁶⁻⁹⁾. In the works ^(10,8) estimates were also obtained in $\widetilde{W}_{2x,t}^{2l,l}$ for solutions of parabolic equations. An analogous theorem for problems with zero initial and boundary conditions was announced by L. N. Slobodetskii also for the case $p \neq 2$ ⁽⁸⁾; however, as the author has communicated, his proof is inaccurate. As for problem (6), an estimate of its solutions for $\rho = 0$ in the space $\widetilde{W}_p^2(\Omega)$ was published by the author of the present article in ⁽¹¹⁾.

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Note: Figure translations are in progress. See original paper for figures.

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