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Abstract

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ON THE UNIQUENESS OF THE SOLUTION OF THE INVERSE PROBLEM OF THE META-HARMONIC POTENTIAL

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The formulation of the inverse problem of potential theory and the first proof of the uniqueness theorem for this problem in the class of star-shaped bodies of constant density belong to P. S. Novikov (^{1}). Further results on this question for bodies with the same constant density, in other formulations, were obtained in the works of L. N. Sretenskii (^{2}), I. T. Todorov and D. Zidarov (^{3}), and others. In the present note, using the methods of ($^{1-3}$), we prove uniqueness of the solution of the inverse problem of the metaharmonic volume potential for bodies of constant density.

By the metaharmonic volume potential of a body T we shall mean the function

$$v(M) = \iiint\limits_T \rho(P) \frac{e^{-\lambda r}}{r} \, d\tau_P,$$

where $\rho(P)$ is the density of the potential, $\lambda = \text{const} < 0$,

$$r = r_{MP} = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}, \qquad d\tau_P = d\xi \, d\eta \, d\zeta.$$

Outside the body T the function $V(M)$ is a regular solution of the metaharmonic equation:

$$L(u) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \lambda^2 u = 0, \tag{1}$$

and inside the body T it is a solution of the equation

$$L(u) = -4\pi\rho(M).$$

Let the bodies T_1 and T_2, with the same constant density $\rho(M) = 1$, occupy bounded simply connected domains in three-dimensional space with boundaries

S_1 and S_2, respectively. Denote by S^i_α $(\alpha = 1, 2)$ the part of the surface S_α internal with respect to $\overline{T_1 \cup T_2}$, and by S^i the sum $S^i_1 \cup S^i_2$. Let S^e be the boundary of the body $\overline{T_1 \cup T_2}$, and $S^e_\alpha = S^e \cap S_\alpha$ the common part of the surfaces S^e and S_α. It is clear that $\overline{T}_\alpha = T_\alpha \cap S_\alpha$, $\overline{T_1 \cup T_2} = T_1 \cup T_2 \cup S^e$. The surfaces S_α $(\alpha = 1, 2)$ are assumed to be piecewise smooth, so that for the domains T_1 and T_2 the known formulas for transforming volume integrals into surface integrals hold. In addition we shall assume that the center of mass of each of the two bodies lies inside the body itself.

Theorem. *If the z-axis (with unit vector $\mathbf{k}$) of the rectangular coordinate system x, y, z can be chosen so that, for the unit vector $\vec{\nu}$ of the exterior normal to S_α, the condition*

$$\int\limits_{S^i} |\cos(\mathbf{k}, \vec{\nu})| \, dS \leq \int\limits_{S^e} |\cos(\mathbf{k}, \vec{\nu})| \, dS \tag{2}$$

is satisfied and, moreover, if the bodies T_1 and T_2 generate identical metaharmonic volume potentials outside $\overline{T_1 \cup T_2}$, then these bodies coincide.

We first prove a lemma (see (^{1})):

Lemma. If the metaharmonic volume potential of some mass with density $\rho(x, y, z)$ is identically equal to zero outside the domain T occupied by the mass, then for any solution $u(x, y, z)$, regular in the domain T, of the metaharmonic equation (1), bounded in the closed domain T, the equality

$$\iiint_T u\rho \, d\tau = 0$$

holds.

Let u' be a regular solution of equation (1) in the ball K with boundary Γ, with continuous partial derivatives at the points of the sphere Γ, which wholly contains within itself the domain occupied by the mass. We apply Green' s formula to the functions u' and the metaharmonic potential $v(M)$:

$$\iiint_T [u' L(v) - v L(u')] \, d\tau = \iint_\Gamma \left(u' \frac{dv}{dv} - v \frac{du'}{dv} \right) ds. \tag{3}$$

In view of the assumptions made concerning u' and v $(v = 0,\ dv/dv = 0$ on $\Gamma,\ L(u') = 0$ inside $K)$, from formula (3) we obtain

$$\iiint_T u' \rho \, d\tau = 0.$$

Any solution $u(x, y, z)$ of equation (1), bounded in the domain T, can be approximated inside T by a solution u' of equation (1) with partial derivatives continuous on the boundary, in the sense that

$$\iiint_T (u - u')\, d\tau$$

can be made arbitrarily small. On this basis we conclude that

$$\iiint_T u\rho\, d\tau = 0.$$

We proceed to the proof of the theorem. First of all, note that, when the exterior metaharmonic volume potentials of the bodies T_1 and T_2 are equal, the centers of mass of both bodies coincide and the masses of these bodies are equal to each other. In view of this, we conclude that the surfaces S_1 and S_2 of both bodies intersect (^{2}).

Let now T denote the domain bounded by the surface S^e. Denote the metaharmonic volume potential of the body T_1 by

$$v_1 = \iiint_{T_1} \rho \frac{e^{-\lambda r}}{r}\, d\tau,$$

and the metaharmonic volume potential of the body T_2 by

$$v_2 = \iiint_{T_2} \rho \frac{e^{-\lambda r}}{r}\, d\tau.$$

Then the function

$$v = v_1 - v_2$$

may be regarded as the metaharmonic volume potential of the body T, if the density $\rho = \rho_1(\xi, \eta, \zeta)$ of the mass occupying the domain T is defined as follows: $\rho_1 = +1$ in those parts of T bounded by the surfaces S_2^i and S_1^e; $\rho_1 = -1$ in those parts of T that are bounded by the surfaces S_2^e and S_1^i; in the remaining parts of T the density $\rho_1 = 0$.

By the condition of the theorem, the exterior metaharmonic volume potentials of the bodies T_1 and T_2 coincide outside $\overline{T}_1 \cup \overline{T}_2$; therefore the body T of density $\rho_1(\xi, \eta, \zeta)$ will have zero exterior metaharmonic volume potential. Then, for any solution u of the metaharmonic equation (1), bounded ···

bounded in the domain $T_1 \cup T_2$, by the indicated lemma, we shall have the equality

$$\int_{\dot{T}_1} u\,d\tau - \int_{\dot{T}_2} u\,d\tau = 0. \tag{4}$$

Now, if $H(x, y, z)$ is a regular solution of the equation in $T_1 \cup T_2$, having continuous partial derivatives in $\overline{T_1 \cup T_2}$, then, putting $u = \partial H/\partial z$, by Ostrogradsky' s formula, from (4) we obtain the equality

$$\int_{S_1} H(x, y, z) \cos(\mathbf{k}, \vec{\nu})\,dS = \int_{S_2} H(x, y, z) \cos(\mathbf{k}, \vec{\nu})\,dS. \tag{5}$$

Equality (5) can be extended to all piecewise-continuous in $\overline{T_1 \cup T_2}$ (metaharmonic in $T_1 \cup T_2$) functions $H(x, y, z)$. To prove the coincidence of the bodies T_1 and T_2, let us construct such a function H for which, from the extended equality (5), we obtain a contradiction with the condition of the theorem.

As $H(x, y, z)$ we take the solution of the Dirichlet problem for equation (1) with the following boundary conditions*

$$H(M) = \begin{cases} -\operatorname{sign}\cos(\mathbf{k}, \vec{\nu}) & \text{for } M \in S_1^e, \\ \operatorname{sign}\cos(\mathbf{k}, \vec{\nu}) & \text{for } M \in S_2^e. \end{cases} \tag{6}$$

For the function $H(M)$ specified by (6), equality (5) takes the form

$$\int_{S_1^e} -\operatorname{sign}\cos(\mathbf{k}, \vec{\nu}) \cdot \cos(\mathbf{k}, \vec{\nu})\,dS + \int_{S_1^i} H \cos(\mathbf{k}, \vec{\nu})\,dS =$$

$$= \int_{S_2^e} \operatorname{sign}\cos(\mathbf{k}, \vec{\nu}) \cdot \cos(\mathbf{k}, \vec{\nu})\,dS + \int_{S_2^i} H \cos(\mathbf{k}, \vec{\nu})\,dS$$

or

$$\int_{S_1^i} H \cos(\mathbf{k}, \vec{\nu})\,dS - \int_{S_2^i} H \cos(\mathbf{k}, \vec{\nu})\,dS = \int_{S^e} |\cos(\mathbf{k}, \vec{\nu})|\,dS.$$

Finally, using the extremum principle for H, the left-hand side of the last equality can be estimated as follows:

$$\int_{S_1^i} H \cos(\mathbf{k}, \vec{\nu})\,dS - \int_{S_2^i} H \cos(\mathbf{k}, \vec{\nu})\,dS < \int_{S^i} |\cos(\mathbf{k}, \vec{\nu})|\,dS.$$

Ultimately we obtain the inequality

$$\int_{S^e} |\cos(\mathbf{k}, \vec{\nu})| \, dS < \int_{S^i} |\cos(\mathbf{k}, \vec{\nu})| \, dS,$$

which contradicts the condition of the theorem.

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References Cited

1. P. S. Novikov, DAN, 18, No. 3, 165 (1938).
2. L. N. Sretenskii, DAN, 99, No. 1, 21 (1954).
3. I. T. Todorov and D. Zidorov, DAN, 120, No. 2, 262 (1958).

* One may assume that the intersection of the surfaces has measure zero. If this is not so, the theorem is easily proved.

Note: Figure translations are in progress. See original paper for figures.