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Abstract

Full Text

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ON CHARACTERISTIC CLASSES FOR FINITE GROUPS

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Let G be a finite group, and let ω be a complex linear representation of the group G. In the present note to the representation ω there is assigned an element $c(\omega) \in H^*(G, \mathbf{Z})$—its characteristic class. This correspondence possesses all the usual properties of characteristic classes. In contrast to the topological situation, the characteristic classes considered here completely determine the representations themselves and are thus a cohomological equivalent of the characters of a finite group.

1. Let $\omega_1, \dots, \omega_k$ be all irreducible complex representations of the finite group G. The module $K(G) = \{n_1\omega_1 + \dots + n_k\omega_k\}$, where the n_i are integers, with respect to the tensor product of representations becomes a commutative ring—the representation ring of the group G. Denote by $K^+(G)$ the semigroup in $K(G)$ consisting of representations (i.e., of elements with $n_i \geq 0$). The operation of taking the exterior power makes $K(G)$ into a λ-ring in the sense of Grothendieck (^{1}). In other words, mappings $\lambda^i : K(G) \to K(G)$ $(i \geq 0)$ are defined in $K(G)$, possessing the properties

$$\lambda^0(x) = 1, \qquad \lambda'(x) = x, \qquad \lambda_i^n(x + y) = \sum_{i+j=n} \lambda^i(x)\lambda^j(y).$$

On $K^+(G)$ the mappings λ^i are given by the formula $\lambda^i(\omega) = \Lambda^i\omega$. If two groups G_1, G_2 and a homomorphism $\varphi : G_1 \to G_2$ are given, then the mapping $\omega_2 \to \varphi^\circ\omega_2$ is naturally extended to a homomorphism of λ-rings $\varphi' : K(G_2) \to K(G_1)$; thus the operation $K(G)$ may be regarded as a contravariant functor from the category of finite groups $\mathfrak{G}$ to the category of λ-rings $\mathfrak{K}$.

We construct one more contravariant functor $\widetilde{A} : \mathfrak{G} \to \mathfrak{K}$. Let $A = \sum_{i \geq 0} A^i$ be a commutative graded ring with identity;

$$\hat{A} = \prod_{i \geq 0} A^i$$

is the ring of formal power series $\sum_{i=0}^{\infty} a_i t^i,\ a_i \in A^i$; $1 + \hat{A}^+$ is the subgroup of the multiplicative group $\hat{A}$, formed by series with $a_0 = 1$ and written additively. Following Grothendieck (^{1}), into the additive group

$$\widetilde{A} = \mathbf{Z} \times (1 + \hat{A}^{+})$$

one can introduce the structure of a commutative λ-ring by means of universal formulas (the same ones that express the characteristic classes of a tensor product and of an exterior power of vector bundles in terms of the Chern classes of the factors). Let G be a finite group; take as

$$A(G) = \sum_{i=0}^{\infty} H^{2i}(G, \mathbf{Z})$$

and denote the resulting contravariant functor $G \to \widetilde{A}(G)$ from $\mathfrak{G}$ to $\mathfrak{K}$ by $\widetilde{A}$.

From what was said above it follows that a reasonable definition of characteristic classes for finite groups must be a natural transformation of functors $K \to \widetilde{A}$.

2. For a finite group G denote by B_G the Eilenberg-MacLane space $K(G, 1)$, i.e. the cell complex with $\pi_1(B_G) = G$ and $\pi_i(B_G) = 0$ for $i > 1$. The universal covering space E_G for B_G may be regarded as a principal skew product with group G. Let $\omega \in K^{+}(G)$; ω may be regarded as a class of homomorphisms $\omega : G \to U(n)$ (relative to inner automorphisms of $U(n)$; $U(n)$ is the unitary group). This homomorphism gives rise to a complex vector n-bundle χ_ω with base B_G. By definition, we shall regard as the characteristic classes for ω the Chern characteristic classes of the bundle χ_ω,

$$c(\omega) = c(\chi_\omega) = \sum_{i=0}^{n} c_i(\chi_\omega), \qquad c_i(\chi_\omega) \in H^{2i}(B_G, \mathbb{Z}) = H^{2i}(G, \mathbb{Z}).$$

Or, otherwise, if by $B_{U(n)}$ we denote the classifying space for the unitary group $U(n)$, then, as is known, $H^*(B_{U(n)}, \mathbb{Z})$ is the polynomial ring in the universal Chern classes y_i, $H^*(B_{U(n)}, \mathbb{Z}) = \mathbb{Z}[y_1, \ldots, y_n]$, $\deg y_i = 2i$. The homomorphism $G \xrightarrow{\omega} U(n)$ gives rise to the restriction homomorphism $H^*(B_{U(n)}, \mathbb{Z}) \xrightarrow{\omega^*} H^*(G, \mathbb{Z})$; then $c_i(\omega) = \omega^* y_i$. Put $\tilde{c}(\omega) = (n, c(\omega)) \in \widetilde{A}(G)$ and extend $\tilde{c}$ in the usual way to all of $K(G)$.

Theorem 1. $\tilde{c}$ is a natural transformation of the functor K into the functor $\widetilde{A}$.

From the formulas

$$\chi_{\omega+\omega'} = \chi_\omega + \chi_{\omega'}, \qquad \chi_{\omega\otimes\omega'} = \chi_\omega \otimes \chi_{\omega'}, \qquad \chi_{\Lambda^p\omega} = \Lambda^p \chi_\omega$$

it follows that

$$\tilde{c}(\omega + \omega') = \tilde{c}(\omega) + \tilde{c}(\omega'), \qquad \tilde{c}(\omega \otimes \omega') = \tilde{c}(\omega) \cdot \tilde{c}(\omega'), \qquad \tilde{c}(\Lambda^p \omega) = \lambda^p \tilde{c}(\omega);$$

this proves that the mapping

$$\tilde{c}_G : \quad K(G) \to \tilde{A}(G) \tag{1}$$

is a homomorphism of λ-rings. The fact that $\tilde{c}_G$ depends functorially on G is verified directly.

Theorem 2. For any finite group G, the homomorphism (1) is a monomorphism.

In other words, two representations ω and ω' of the same degree n coincide if their characteristic classes are equal: $c(\omega) = c(\omega')$. For cyclic groups this is verified by direct calculation; for general groups it follows from the existence of characters.

If ω^* is the complex-conjugate representation to ω, then

$$c_i(\omega^*) \simeq (-1)^i c_i(\omega),$$

therefore for real representations ω,

$$2c_{2k+1}(\omega) = 0.$$

In particular, if G is a group of odd order and ω is a real representation, then

$$c_{2k+1}(\omega) = 0.$$

3. This section is devoted to the computation of the characteristic classes modulo a prime for the regular representation.

Let G be a group of order n; let p be a fixed prime divisor of n;

$$\omega = \omega(G)$$

the regular representation of the group G;

$$c(G) = \sum_{i=0}^{n} c_i(\omega)$$

the characteristic class of ω; $c(G)_p$ the characteristic class of ω modulo p, i.e. the image of $c(G)$ under the homomorphism i_*, induced by the coefficient homomorphism

$$\mathbb{Z} \xrightarrow{i} \mathbb{Z}_p;$$

$P^l : H^n(G, \mathbb{Z}_p) \to H^{n+2l(p-1)}((G, \mathbb{Z})_p)$ are the reduced Steenrod powers. To compute the class $c(G)_p$ it is enough to restrict to the case of a p-group; indeed, let π be a Sylow p-subgroup of the group G, $n = p^{kq}$, $(p, q) = 1$, $\operatorname{ord} \pi = p^k$; the restriction homomorphism

$$H^*(G, \mathbb{Z}_p) \to H^*(\pi, \mathbb{Z}_p)$$

is an embedding and, by the naturality of characteristic classes,

$$c(G)_p = (c(\pi)_p)^q.$$

Theorem 3. Let G be a p-group of order $n = p^k > 2$;

$$c(G) = \sum_{i=0}^{n} c_i(G)$$

is the characteristic class of the regular representation of G modulo p. Then $c_i(G) = 0$ except, possibly, for the following dimensions:

$$c_{p^k-p^{k-1}}(G),\ c_{p^k-p^{k-2}}(G), \dots,\ c_{p^k-1}(G);$$

moreover,

$$c_{p^k-p^s}(G) = P^l(c_{p^k-p^{k-1}}(G)), \quad \text{where } l = p^s(1 + p + \dots + p^{k-s-2}).$$

The proof of the theorem is based on two lemmas.

Lemma 1. If $G = Z_p$, then

$$c(Z_p)_p = 1 - \xi^{p-1},$$

where ξ is a generator of the group $H^2(Z_p, Z_p)$. If $G = G_1 \times Z_p$, $n = \operatorname{ord} G_1$,

$$c(G_1)_p = 1 + \omega_1 + \dots + \omega_n,$$

then

$$c(G)_p = (1 - \xi^{p-1})^n + \sigma_1 (1 - \xi^{p-1})^{n-1} + \dots + \sigma_n,$$

where

$$\sigma_l = P^l(\omega_l) - c_l^1 \xi^{p-1} P^{l-1}(\omega_l) + \dots + (-1)^l \xi^{l(p-1)} P^0(\omega_l).$$

The lemma follows from the explicit formulas for the characteristic classes of the tensor product of representations and from the explicit expressions for the Steenrod operations on characteristic classes (2).

The next lemma is purely combinatorial in nature.

Lemma 2. Let $\sigma_1, \dots, \sigma_n$ be the elementary symmetric functions in the variables $x_1, \dots, x_n$, and let $F(x_1, \dots, x_n)$ be a symmetric polynomial with typical term

$$x_1^p \cdots x_k^p x_{k+1} \cdots x_i;$$

then

$$F(x_1, \dots, x_n) = A\sigma_j + f(\sigma_1, \dots, \sigma_{j-1}),$$

where

$$A = \pm j \frac{(i-1)!}{(i-k)!\,k!}, \quad j = (p-1)k + i.$$

Let us indicate the proof of the theorem. From Lemma 2 and the Borel-Serre formulas (^{2}) for the reduced Steenrod powers of characteristic classes it follows that, in order to prove the theorem, it is enough to prove that $c_i(G)_p = 0$ for $i < p^k - p^{k-1}$. Lemma 1 allows this fact to be reduced to the case when G has the form $G = G_1 \times Z_p$. Let ξ be a one-dimensional representation of G induced by a representation of Z_p; let ω be the regular representation of G. Then $\xi \otimes \omega = \omega$, and the assertion follows from the explicit formulas for the characteristic class in the left-hand side of the equality and from the algebraic independence of the generator $\xi \in H^2(Z_p, Z_p) \subset H^2(G, Z_p)$ over the algebra $H^*(G_1, Z_p)$.

Thus, if G is a finite group of order p^{kq}, $(p, q) = 1$, then the first nontrivial characteristic class modulo p in $c(G)$ has dimension $(p^k - p^{k-1})q$. It is easy to see that it is always nonzero and even has infinite order (multiplicative). If H is a subgroup of order p^{lt}, then

$$i(G, H) c_{(p^k - p^{k-1})q}(G)_p = \left(c_{(p^l - p^{l-1})t}(H)_p\right)^{(G:H)};$$

in particular, the following holds.

Corollary 1. Let G be a group of order p^{kq}, $(p, q) = 1$; then for any subgroup H the restriction homomorphism

$$i(G : H) : H^{(p^k - p^{k-1})q}(G, Z_p) \to H^{(p^k - p^{k-1})q}(H, Z_p)$$

is nonzero.

It is easy to see that the index $(p^k - p^{k-1})q$ cannot, in general, be decreased.

Characteristic classes are also useful for computing the cohomology of homogeneous spaces $U(n)/G$, where G is a finite group. We indicate a simple result of this kind.

Corollary 2. Let $G \subset U(n)$ be a finite subgroup of order $n = p^k q$ of the unitary group, corresponding to the regular representation; then, up to dimension $2(p^k - p^{k-1})q - 2$,

$$H^*(U(n)/G, \mathbf{Z}_p) \approx H^*(U(n), \mathbf{Z}_p) \otimes H^*(G, \mathbf{Z}_p).$$

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REFERENCES

1. A. Grothendieck, Bull. Soc. math. France, **86**, 2, 137 (1958).
2. *Fiber spaces and their applications*, Moscow, 1958, pp. 247—281.

Note: Figure translations are in progress. See original paper for figures.