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Abstract

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A PHRAGMÉN-LINDELÖF TYPE THEOREM FOR HARMONIC FUNCTIONS IN A RECTANGULAR CYLINDER

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Theorem 1. $u(x, y, t)$ is a harmonic function in the cylinder over the rectangle $0 \leq x \leq a,\ 0 \leq y \leq b,\ -\infty < t < \infty$. If the conditions

$$u(x, y, t)\big|_{\Gamma} = 0, \quad \text{where } \Gamma \text{ is the surface of the cylinder;} \tag{1}$$

$$|\partial u(x, 0, t)/\partial y| < c; \qquad |\partial u(x, b, t)/\partial y| < c; \tag{2}$$

$$\max_{(x,y)} |u(x, y, t)| < c \exp e^{\pi|t|/(b+\varepsilon)}, \qquad \varepsilon > 0, \tag{3}$$

are fulfilled, then $u(x, y, t) \equiv 0$.

The proof of the theorem is based on the uniqueness theorem for Dirichlet series ([1]) and repeats the main features of the proof of Theorem 1 of the same work. A solution satisfying condition (1) is represented in the form of a series in eigenfunctions

$$v(x, y, t) = \sum_{n,m} \{a_{n,m} \exp[\pi\lambda_{n,m} t] + b_{n,m} \exp[-\pi\lambda_{n,m} t]\} \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b},$$

where

$$\lambda_{n,m} = \sqrt{n^2/a^2 + m^2/b^2}.$$

It is sufficient to consider only the sum

$$u(x, y, t) = \sum_{n,m} a_{n,m} \exp[\pi\lambda_{n,m} t] \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}, \tag{4}$$

since for it too the conditions (1), (2), (3) of the theorem must be fulfilled if they are fulfilled for $v(x, y, t)$.

Condition (2) for $u(x, y, t)$ leads to the condition of boundedness on the real axis $-\infty < t < \infty$ of the following Dirichlet series:

$$f_n(z) = \sum_{k=0}^{\infty} (2k+1) a_{n,2k+1} \exp[\pi \lambda_{n,2k+1} z],$$

$$\varphi_n(z) = \sum_{k=1}^{\infty} 2k a_{n,2k} \exp[\pi \lambda_{n,2k} z].$$

Indeed, let us write condition (2) explicitly:

$$\left| \frac{\partial u(x,0,t)}{\partial y} \right| = \left| \frac{\pi}{b} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \sum_{m=1}^{\infty} m a_{n,m} \exp[\pi \lambda_{n,m} t] \right| < c; \tag{5}$$

$$\left| \frac{\partial u(x,b,t)}{\partial y} \right| = \left| \frac{\pi}{b} \sum_{n=1}^{\infty} \sin \frac{n\pi x}{a} \sum_{m=1}^{\infty} m(-1)^m a_{n,m} \exp[\pi \lambda_{n,m} t] \right| < c. \tag{6}$$

Using the orthogonality of the functions $\sin(n\pi x/a)$, $n = 1, 2, \ldots$, on the interval $[0, a]$, from conditions (5), (6) one can obtain boundedness of the sums

$$\left| \sum_{m=1}^{\infty} m a_{n,m} \exp[\pi \lambda_{n,m} t] \right| < c; \qquad \left| \sum_{m=1}^{\infty} m(-1)^m a_{n,m} \exp[\pi \lambda_{n,m} t] \right| < c.$$

Adding and subtracting these sums, we obtain that $f_n(z)$ and $\varphi_n(z)$ are bounded on the real axis. The uniqueness theorem ([1]) is applicable to $f_n(z)$ and $\varphi_n(z)$, and therefore all $a_{n,m} = 0,\ n, m = 1, 2, \ldots$. The theorem is proved.

It should be noted that the theorem uses the condition of boundedness of the normal derivative only on two opposite faces of the rectangular cylinder. One can construct an example of a harmonic function having growth constant π/b, i.e., a function for which

$$\max_{(x,y)} |u(x, y, t)| = \exp \psi(t) e^{\pi(t)/b},$$

where $|\psi(t)| < c,\ -\infty < t < \infty$, whose normal derivative is bounded on the sides of the cylinder of width a, and also an example of a function with growth constant π/a and bounded normal derivative on the sides of width b. The construction of examples of such functions proves the sharpness of Theorem 1. However, it remained unclear whether the constant π/b (we assume $b \leq a$) increases if boundedness of the normal derivative is required on the entire boundary of the cylinder. We shall prove the following theorem:

Theorem 2. There exists a harmonic function $u(x, y, t) \not\equiv 0$ in the cylinder $0 \le x \le a,\ 0 \le y \le b,\ -\infty < t < \infty$ $(b \le a)$, satisfying the conditions

$$u(x,y,t)\big|_{\Gamma} = 0, \qquad \Gamma \text{ is the surface of the cylinder;} \tag{1'}$$

$$|\partial u(x,y,t)/\partial x|\,\Big|_{\substack{x=0\\x=a}} < c; \qquad |\partial u(x,y,t)/\partial y|\,\Big|_{\substack{y=0\\y=b}} < c; \tag{2'}$$

$$\max_{(x,y)} |u(x,y,t)| \le \exp c e^{\pi|t|/b}. \tag{3'}$$

Denote

$$v_\alpha(x,y,t) = \frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2} \frac{\sin \pi x w}{\cos^{1/2}\pi a w}\,\frac{\sin \pi y u}{\cos^{1/2}\pi b u}\times$$

$$\times\frac{\exp\left[-\alpha\sqrt{u^2+w^2}\ln\sqrt{u^2+w^2}+\sqrt{u^2+w^2}\,\pi t\right]}{\left(\sqrt{u^2+w^2}+1\right)^2\sqrt{u^2+w^2}}\,du\,dw. \tag{7}$$

The contour L_1 consists of the part of the circle $|w| = \varepsilon,\ |\arg w| \le \pi/8$ and two rays $\arg w = \pm\pi/8,\ \varepsilon \le |w| < \infty$. The contour L_2 is defined analogously: $|u| = \varepsilon,\ |\arg u| \le \pi/8;\ \arg u = \pm\pi/8,\ \varepsilon \le |u| < \infty;\ \varepsilon > 0$.

Lemma 1. The function $v_\alpha(x, y, t)$, defined by the integral (7), is a harmonic function in the domain

$$0 \le x \le a/2, \qquad 0 \le y \le b/2, \qquad -A \le t \le A \quad (A \text{ arbitrary}) \tag{8}$$

and is equal, by the residue theorem, $\lambda_{n,m} = \sqrt{(2n+1)^2/a^2 + (2m+1)^2/a^2}$,

$$v_\alpha(x,y,t) = \sum_{n,m=0}^{\infty} \frac{\exp[-\alpha\lambda_{n,m}\ln\lambda_{n,m}]}{(\lambda_{n,m}+1)^2\lambda_{n,m}}\times$$

$$\times(-1)^{n+m}\exp[\pi\lambda_{n,m}t]\,\sin\frac{(2n+1)\pi x}{a}\,\sin\frac{(2m+1)\pi y}{a}. \tag{9}$$

Let us establish the absolute and uniform convergence in the domain (8) of the integral (7). The contours L_1 and L_2 are chosen so that on them the expression $u^2 + w^2$ never vanishes. Moreover, when $w \in L_1,\ u \in L_2$, the estimates

$$\left|\arg\sqrt{u^2+w^2}\right| < \pi/4; \qquad \sqrt[4]{|u|^4+|w|^4} \le \left|\sqrt{u^2+w^2}\right| \le \sqrt{|u|^2+|w|^2}. \tag{10}$$

hold. Therefore

$$\left|\frac{\sin \pi x w}{\cos^{1/2}\pi a w}\,\frac{\sin \pi y u}{\cos^{1/2}\pi b u}\,\frac{\exp\left[-\alpha\sqrt{u^2+w^2}\ln\sqrt{u^2+w^2}+\pi t\sqrt{u^2+w^2}\right]}{\left(\sqrt{u^2+w^2}+1\right)^2\sqrt{u^2+w^2}}\right| <$$

$$< \frac{\exp\left[-(\alpha/\sqrt{2})\sqrt[4]{|u|^4+|w|^4}\ln\sqrt[4]{|u|^4+|w|^4}+(\pi/4+\pi A)\sqrt{|u|^2+|w|^2}\right]}{\sqrt[4]{|u|^4+|w|^4}\left(\sqrt[4]{|u|^4+|w|^4}+1\right)^2}.$$

The harmonicity of the function $v_\alpha(x, y, t)$ is verified directly by differentiating with respect to the parameters x, y, t. From the estimates (10) it also follows that the integral (7) can be computed as the sum of residues at the simple poles

$$w = (2n+1)/a, \quad n = 0, 1, \dots, \qquad u = (2m+1)/b, \quad m = 0, 1, \dots .$$

Lemma 2. If the parameter α in (7) is chosen so that $\alpha = \min(a, b)$, then in the domain (8) the conditions

$$|\partial v_\alpha(0, y, t)/\partial x| < c; \qquad |\partial v_\alpha(x, 0, t)/\partial y| < c \tag{11}$$

will be satisfied.

By the results of Lemma 1, differentiation of the integral (7) with respect to the parameters x, y is possible. We have:

$$\frac{\partial v_\alpha(x,0,t)}{\partial y} = \frac{1}{(2\pi i)^2} \iint_{L_1 L_2} \frac{u}{\cos^{1/2}\pi bu}\,\frac{\sin \pi xw}{\cos^{1/2}\pi aw} \times \frac{\exp\left[-\alpha\sqrt{u^2+w^2}\ln\sqrt{u^2+w^2} + \pi t\sqrt{u^2+w^2}\right]}{\left(\sqrt{u^2+w^2}+1\right)^2\sqrt{u^2+w^2}}\,du\,dw; \tag{12}$$

$$\frac{\partial v_\alpha(0,y,t)}{\partial x} = \frac{1}{(2\pi i)^2} \iint_{L_1 L_2} \frac{w}{\cos^{1/2}\pi aw}\,\frac{\sin \pi yu}{\cos^{1/2}\pi bu} \times \frac{\exp\left[-\alpha\sqrt{u^2+w^2}\ln\sqrt{u^2+w^2} + \pi t\sqrt{u^2+w^2}\right]}{\left(\sqrt{u^2+w^2}+1\right)^2\sqrt{u^2+w^2}}\,du\,dw. \tag{13}$$

The integrals obtained are absolutely convergent and, moreover, the contours may be closed at infinity.

Transform the integral (12). By the residue theorem we have

$$\begin{aligned}\frac{\partial v_\alpha(x,0,t)}{\partial y} &= \frac{1}{2\pi i}\int_{L_2}\sum_{n=0}^{\infty}(-1)^n \sin\frac{(2n+1)\pi x}{a}\,\frac{u}{\cos^{1/2}\pi bu}\times \\ &\times \frac{\exp\left[-\alpha\sqrt{u^2+(2n+1)^2/a^2}\ln\sqrt{u^2+(2n+1)^2/a^2} + \pi t\sqrt{u^2+(2n+1)^2/a^2}\right]}{\left(\sqrt{u^2+(2n+1)^2/a^2}+1\right)^2\left(\sqrt{u^2+(2n+1)^2/a^2}\right)}\,du \\ &= \sum_{n=0}^{\infty}(-1)^n\sin\frac{(2n+1)\pi x}{a}\,f_n(t),\end{aligned} \tag{14}$$

where

$$f_n(t) = \frac{1}{2\pi i}\int_{L_2} \frac{u}{\cos^{1/2}\pi b u} \times \frac{\exp\left[-\alpha\sqrt{u^2+(2n+1)^2/a^2}\ln\sqrt{u^2+(2n+1)^2/a^2}+\pi t\sqrt{u^2+(2n+1)^2/a^2}\right]}{\left(\sqrt{u^2+(2n+1)^2/a^2}+1\right)^2\sqrt{u^2+(2n+1)^2/a^2}}\,du.$$

Put $z = \sqrt{u^2+(2n+1)^2/a^2}$, $du = z\,dz/\sqrt{z^2-(2n+1)^2/a^2}$,

$$f_n(t) = \frac{1}{2\pi i}\int_{L_2^{(n)}} \frac{\exp[-\alpha z\ln z+\pi t z]}{\cos^{1/2}\pi b\sqrt{z^2-(2n+1)^2/a^2}\,(z+1)^2}\,dz. \qquad (15)$$

It is clear that the deformed contour $L_2^{(n)}$ lies inside the domain $\operatorname{Re} z > 0$. The integrand in (15) has, in the domain bounded by $L_2^{(n)}$, and in the domain $\operatorname{Re} z \geq 0$, the same singular points; moreover, it decreases well as $|z| \to \infty$, $\operatorname{Re} z \geq 0$, so that the contour of integration may be shifted to the imaginary axis. Estimate $f_n(t)$. Suppose that $b \leq a$ and $\alpha = b$;

$$f_n(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\exp[-bz\ln z+\pi t z]}{\cos^{1/2}\pi b\sqrt{z^2-(2n+1)^2/a^2}\,(z+1)^2}\,dz,$$

$$\begin{aligned}|f_n(t)| &< c\int_0^\infty \frac{\exp\left[-\frac{1}{2}\pi b\sqrt{y^2+(2n+1)^2/a^2}+\frac{1}{2}\pi b y\right]}{(y+1)^2}\,dy \\ &= c\left\{\int_0^{(2n+1)^{3/2}}\cdots+\int_{(2n+1)^{3/2}}^\infty\cdots\right\} < \\ &< ce^{-c\sqrt{2n+1}}\int_0^{(2n+1)^{3/2}}\frac{dy}{(y+1)^2}+c\int_{(2n+1)^{3/2}}^\infty\frac{dy}{(1+y)^2} < \frac{c}{(2n+1)^{3/2}},\end{aligned} \qquad (16)$$

since, for $0 \leq y \leq (2n+1)^{3/2}$,

$$-\sqrt{y^2+(2n+1)^2/a^2}+y = -(2n+1)^2/a^2\left(y+\sqrt{y^2+(2n+1)^2/a^2}\right) < -c\sqrt{2n+1}.$$

From (14) and the estimate (16) it follows that $|\partial v_a(x,0,t)/\partial y| < c$. Analogously one can show that $|\partial v_a(0,y,t)/\partial x| < c$ for $a \geq b$ and $\alpha = b$.

Proof of Theorem 2. Formula (9) extends the harmonic function $v_\alpha(x,y,t)$, defined in the cylinder $0 \leq x \leq a/2$, $0 \leq y \leq b/2$, $-\infty < t < \infty$, to the cylinder $0 \leq x \leq a$, $0 \leq y \leq b$, $-\infty < t < \infty$. In this case

$$v_\alpha(x,y,t) = v_\alpha(a-x,y,t) = v_\alpha(x,b-y,t) = v_\alpha(a-x,b-y,t), \qquad (17)$$

i.e., this function takes equal values at points symmetric with respect to the midlines of the rectangle. Put now $\alpha = b$. Denote $v_b(x, y, t) = u(x, y, t)$. The function $u(x, y, t)$ satisfies the conditions of Lemma 2 and, by virtue of property (17), has a bounded derivative already on the entire surface of the cylinder. Let us estimate the growth of this function inside the cylinder. Let

$$\lambda_{n,m} = \sqrt{(2n+1)^2/a^2 + (2m+1)^2/b^2},$$

$$\max_{(x,y)} |u(x,y,t)| \leq \sum_{n,m=0}^{\infty} \frac{\exp[-b\lambda_{n,m} \ln \lambda_{n,m}]}{(\lambda_{n,m}+1)^2 \lambda_{n,m}} \exp[\pi \lambda_{n,m} t] = u\left(\frac{a}{2}, \frac{b}{2}, t\right),$$

$$u\left(\frac{a}{2}, \frac{b}{2}, t\right) < c \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sum_{m=0}^{\infty} \exp[-b\lambda_{n,m} \ln \lambda_{n,m} + \lambda_{n,m} \pi t].$$

It is clear that, as $t \to \infty$,

$$\ln \sum_{m=0}^{\infty} \exp[-b\lambda_{n,m} \ln \lambda_{n,m} + \lambda_{n,m} \pi t] \sim \max_m (-b\lambda_{n,m} \ln \lambda_{n,m} + \lambda_{n,m} \pi t).$$

Calculating this maximum, we obtain that

$$u(a/2, b/2, t) < \exp\{c \exp[\pi t/b]\}.$$

Theorem 2 is proved.

For an infinite strip, i.e., a domain of the form

$$-\infty < x < \infty, \qquad -b/2 \leq y \leq b/2, \qquad -\infty < t < \infty \tag{18}$$

the assertion of Theorem 1 evidently remains valid.

The following example of a harmonic function for the strip (18) is, in meaning, analogous to the proof of Theorem 2 for the rectangular cylinder; it establishes the sharpness of Theorem 1:

$$u(x, y, t) = \mathrm{Re}\{\exp[e^{\pi z/b} - \pi z/b]\}, \quad \text{where } z = t + iy.$$

Comparing the results of Theorem 1 [(1)] with the results of this note, we see that for an infinite strip and for circular and arbitrary rectangular cylinders inscribed tangent to the strip, the growth constant is one and the same and depends only on the width of the strip. These results, being a certain answer to the general formulation of the problem given by S. N. Mergelyan in [2], indicate the presence of purely spatial regularities for theorems of the Phragmén–Lindelöf type.

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References

[1] M. A. Evgrafov, I. A. Chegis, *DAN*, **134**, No. 2 (1960). [2] S. N. Mergelyan, *UMN*, **11**, 5 (71), 3 (1956).

Note: Figure translations are in progress. See original paper for figures.