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Abstract

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N. P. KUPTSOV

ON CONDITIONS FOR NON-SELF-ADJOINTNESS OF A LINEAR DIFFERENTIAL OPERATOR OF SECOND ORDER

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Consider on the half-axis $[0, \infty)$ the operator Ly, defined by the differential expression

$$-y'' + q(x)y \tag{1}$$

and the boundary condition

$$\cos\alpha \cdot y(0) + \sin\alpha \cdot y'(0) = 0. \tag{2}$$

We shall assume the function $q(x)$ to be real and summable on every finite interval belonging to $[0, \infty)$. It is known (([1], pp. 103–109)) that this operator is either self-adjoint (the defect indices in this case are $(0, 0)$) or has defect indices $(1, 1)$. One of the strongest sufficient conditions for self-adjointness of Ly was indicated by Sears ([3]): if there exists a positive nonincreasing function $Q(x)$ such that

$$q(x) \geq -Q(x) \quad \text{and} \quad \int_0^\infty \frac{dx}{\sqrt{Q(x)}} = +\infty,$$

then Ly is a self-adjoint operator.

Some new results in this direction were obtained by Brink ([6]). M. A. Naimark ([2], p. 270) and V. B. Lidskii ([4]) indicated sufficient conditions for non-self-adjointness of Ly:

1°. **M. A. Naimark' s criterion.** If $q(x)$ and $q'(x)$ are absolutely continuous on $[0, \infty)$, $q'(x) \leq 0$, $q''(x) \leq 0$, $\lim_{x\to\infty} q(x) = -\infty$, $|q'(x)| =$

$$= O(|q(x)|^\alpha) \quad \left(0 < \alpha < \frac{3}{2}\right) \quad \text{and} \quad \int_0^\infty \frac{dx}{\sqrt{|q(x)|}} < \infty,$$

then Ly has defect indices $(1,1)$.

2°. **V. B. Lidskii' s criterion***. The operator Ly is non-self-adjoint if $q(x)$ is negative and absolutely continuous on $[0,\infty)$, and

$$q'(x) \leq \frac{2+\varepsilon}{x}\, q(x)$$

for some $\varepsilon > 0$.

In the present note new sufficient conditions for non-self-adjointness of Ly are established.

Let $s(x) > 0$ and $\varphi(x)$ be arbitrary functions absolutely continuous on $[0,\infty)$. Then for any solution of the differential equation

$$y'' - q(x)y = 0$$

the estimate is valid

$$y^2(x) \leqslant \frac{C}{\sqrt{s(x)}} \exp\left\{ \int_0^x \sqrt{\left(\frac{s'}{2s} - 2\varphi\right)^2 + \frac{1}{s}(\varphi' - \varphi^2 + q + s)^2}\, d\xi \right\},$$

where

$$C = \frac{1}{\sqrt{s(0)}} \left[(\varphi^2(0) + s(0))y^2(0) + 2\varphi(0)y(0)y'(0) + {y'}^2(0) \right].$$

Hence it follows:

Theorem 1. If there exist functions $s(x) > 0$ and $\varphi(x)$, absolutely continuous on $[0,\infty)$, such that

$$\int_0^\infty \frac{1}{\sqrt{s(x)}} \exp\left\{ \int_0^x \sqrt{\left(\frac{s'}{2s} - 2\varphi\right)^2 + \frac{1}{s}(\varphi' - \varphi^2 + q + s)^2}\, d\xi \right\} dx < \infty, \quad (3)$$

then the operator Ly has deficiency indices $(1,1)$.

Unfortunately, this criterion is not sufficiently effective, since it is not clear how $s(x)$ and $\varphi(x)$ should be chosen. In this respect the following criterion is more convenient:

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* V. B. Lidskii's criterion was established for systems of the form $\ddot{x}_i = \sum_{j=1}^{n} q_{ij}x_j$. Here a particular criterion is formulated for $n = 1$.

Theorem 2. Let $Q(x)$ be an arbitrary positive function on $[0,\infty)$, absolutely continuous together with $Q'(x)$. If there exists a negative, absolutely continuous function $\bar{q}(x)$ on $[0,\infty)$, possessing the properties (x_0 is an arbitrary positive number):

$$\frac{\bar{q}'(x)}{\bar{q}(x)} \geqslant \frac{Q'(x)}{Q(x)} \quad \text{for } x \geqslant x_0; \tag{4}$$

$$\int_{x_0}^{\infty} \frac{|q(x)-\bar{q}(x)|}{\sqrt{|\bar{q}(x)|}}\,dx < \infty; \tag{5}$$

$$\int_{x_0}^{\infty} \frac{1}{\sqrt{Q(x)}} \exp\left[\frac{1}{4}\int_{x_0}^{x} \frac{1}{\sqrt{|\bar{q}(\xi)|}}\left|\frac{Q''(\xi)}{Q(\xi)} - \frac{5}{4}\frac{{Q'}^2(\xi)}{Q^2(\xi)}\right| d\xi\right] dx < \infty, \tag{6}$$

then the operator Ly has deficiency indices $(1,1)$.

For the proof it is sufficient to use Theorem 1, putting $s(x) = -\bar{q}(x)$ and $\varphi(x) = Q'(x)/4Q(x)$. Then

$$\int_{x_0}^{x} \sqrt{\left(\frac{s'}{2s} - 2\varphi\right)^2 + \frac{1}{s}(\varphi' - \varphi^2 + q + s)^2}\,d\xi \leqslant$$

$$\leqslant \int_{x_0}^{x} \left|\frac{\bar{q}'(\rho)}{2\bar{q}(\xi)} - \frac{Q'(\xi)}{2Q(\xi)}\right| d\xi + \frac{1}{4}\int_{x_0}^{x} \frac{1}{\sqrt{|\bar{q}(\xi)|}}\left|\frac{Q''(\xi)}{Q(\xi)} - \frac{5}{4}\frac{{Q'}^2(\xi)}{Q^2(\xi)}\right| d\xi +$$

$$+\int_{x_0}^{x} \frac{1}{\sqrt{|\bar{q}(\xi)|}}|q(\xi)-\bar{q}(\xi)|\,d\xi = \frac{1}{2}\ln\left|\frac{\bar{q}(x)}{q(0)}\right| - \frac{1}{2}\ln\frac{Q(x)}{Q(0)} +$$

$$+\frac{1}{4}\int_{x_0}^{x} \frac{1}{\sqrt{|\bar{q}(\xi)|}}\left|\frac{Q''}{Q(\xi)} - \frac{5}{4}\frac{{Q'}^2(\xi)}{Q^2(\xi)}\right| d\xi + \int_{x_0}^{x} \frac{|q(\xi)-\bar{q}(\xi)|}{\sqrt{|\bar{q}(\xi)|}}\,d\xi.$$

Now, relying on the conditions of Theorem 2, it is easy to establish the validity of relation (3). From Theorem 2 it is not difficult to obtain the above-mentioned criteria of V. B. Lidskii and M. A. Naimark. In the first case it is sufficient to put $Q(x) = x^{2+\varepsilon}$ and $\bar{q}(x) = q(x)$. To derive M. A. Naimark' s criterion, put $Q(x) = -q(x)$ and $\bar{q}(x) = q(x)$. Then conditions (4)

and (5) will be satisfied. Let us proceed to verify condition (6):

$$\begin{aligned}I(x) &= \int_{x_0}^{x} \frac{1}{\sqrt{|q|}} \left| \frac{q''}{q} - \frac{5}{4}\frac{{q'}^2}{q^2} \right| d\xi \le \int_{x_0}^{x} |q|^{-3/2}|q''|\, d\xi + \frac{5}{4}\int_{x_0}^{x} |q|^{-5/2}{q'}^2\, d\xi \\ &= -q'(x)|q(x)|^{-3/2} + q'(x_0)|q(x_0)|^{-3/2} + \frac{11}{4}\int_{x_0}^{x} |q|^{-5/2}{q'}^2\, d\xi \le \\ &\le M|q(x)|^{\alpha-3/2} + \frac{11M}{4}\int_{x_0}^{x} |q|^{\alpha-5/2}|q'|\, d\xi = \\ &= M|q(x)|^{\alpha-3/2} + \frac{11M}{4(3/2-\alpha)}\left[\, |q(x_0)|^{\alpha-3/2} - |q(x)|^{\alpha-3/2}\,\right],\end{aligned}$$

where

$$M = \sup_{x_0 \le x < \infty} \frac{|q'(x)|}{|q(x)|^{\alpha}}.$$

Taking into account the relation $\alpha < 3/2$, we easily obtain $I(x) \le M_1$, where M_1 does not depend on x. Consequently,

$$\int_{x_0}^{\infty} \frac{1}{\sqrt{Q(x)}} e^{\frac{1}{4}I(x)}\, dx \le e^{\frac{1}{4}M_1} \int_{x_0}^{\infty} \frac{dx}{\sqrt{|q(x)|}} < \infty.$$

Theorem 3. Let $Q(x)$ be an arbitrary positive, nondecreasing, absolutely continuous function on $[0,\infty)$. Suppose, further, that $Q'(x)$ is absolutely continuous and that $Q^{-\beta}(x)$ is convex on $[0,\infty)$ for some $\beta \in (0,1/2)$.

Finally, suppose

$$\int_0^{\infty} \frac{dx}{\sqrt{Q(x)}} < \infty.$$

If there exists a negative absolutely continuous function $\overline{q}(x)$ on $[0,\infty)$, possessing the following properties (x_0 is an arbitrary positive number):

$$\frac{\overline{q}'(x)}{\overline{q}(x)} \ge \frac{Q'(x)}{Q(x)} \quad \text{for } x \ge x_0; \tag{7}$$

$$\int_{x_0}^{\infty} \frac{|q(x) - \overline{q}(x)|}{\sqrt{|\overline{q}(x)|}}\, dx < \infty, \tag{8}$$

then Ly has deficiency indices $(1,1)$.

Proof. It is enough to verify that condition (6) is fulfilled for $Q(x)$. From (7) it follows that

$$|\overline{q}(x)| \geq CQ(x), \tag{9}$$

where $C > 0$ and does not depend on x. Put $Q^{-\beta}(x) = \gamma(x)$. From the properties of $Q(x)$ it follows that $\gamma'(x) \leq 0$ on $[0, \infty)$ and $\gamma''(x) \geq 0$ almost everywhere on $[0, \infty)$. Then, taking (9) into account, we may write

$$K(x) = \int_0^\infty \frac{1}{\sqrt{|\overline{q}(x)|}} \left| \frac{Q''}{Q} - \frac{5}{4} \frac{{Q'}^2}{Q^2} \right| dx \leq$$

$$\leq \frac{1}{\beta\sqrt{C}} \int_0^\infty \gamma^{1/2\beta}(x) \left| \frac{\gamma''(x)}{\gamma(x)} + \left(\frac{1}{4\beta} - 1 \right) \frac{\gamma {a'}^2(x)}{\gamma^2(x)} \right| dx \leq$$

$$\leq C_1 \int_0^\infty \gamma^{1/2\beta - 1} \gamma'' \, dx + C_2 \int_0^\infty \gamma^{1/2\beta-2} \gamma {a'}^2 \, dx,$$

where C_1 and C_2 do not depend on x.

Taking into account the inequality $\beta < 1/2$, we integrate by parts (C_3 is a constant independent of x):

$$K(x) \leqslant C_1 \left[\gamma^{1/2\beta-1}\gamma' \Big|_0^\infty - \left(\frac{1}{2\beta} - 1 \right) \int_0^\infty \gamma^{1/2\beta-2} \gamma {a'}^2 \, dx \right] +$$

$$+ C_2 \left[\gamma^{1/2\beta-1}\gamma' \Big|_0^\infty - \frac{1}{(1/2\beta - 1)} \int_0^\infty \gamma^{1/2\beta-1} \gamma'' \, dx \right] \leqslant C_3.$$

The theorem is proved.

It follows from the work of É. Shnol' (^{5}) that in the formulation of Theorem 3 one cannot omit the convexity of the function $Q^{-\beta}(x)$. From the same work it follows that the differential inequality (7) cannot be replaced by the inequality $\overline{q}(x) \leqslant -Q(x)$.

Saratov State University
named after N. G. Chernyshevsky

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Note: Figure translations are in progress. See original paper for figures.

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