



---

Soviet-era science, translated into English

# MATHEMATICS

1961

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196101.60232>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

**MATHEMATICS**

**G. D. SUVOROV**

**THE LENGTH AND AREA PRINCIPLE FOR  
Q-QUASICONFORMAL MAPPINGS**

*(Presented by Academician M. A. Lavrent'ev, 30 V 1961)*

In the theory of analytic functions, wide-ranging applications have been found for an inequality expressing the so-called "length and area principle" (<sup>1</sup>, p. 26).

In the present note we formulate a generalization of this inequality to the case of non-univalent  $Q$ -quasiconformal mappings of arbitrary plane domains (Theorem 1), and as examples give two of its simplest applications. Theorem 3, which in the final analysis is also a consequence of the basic inequality, is a generalization of Ahlfors' boundary distortion theorem for conformal mappings (<sup>2</sup>, p. 96) to the case of univalent  $Q$ -quasiconformal mappings.

1°. Let, in an arbitrary (not necessarily simply connected) domain  $D$  of the plane  $z = x + iy$ , a continuous distribution of characteristics  $(q(z), \theta(z))$  be given, and let  $w = T(z)$  be an interior (in the sense of C. Stoilow) quasiconformal mapping of the domain  $D$  into the plane  $w = u + iv$  (see (<sup>3</sup>, p. 24)) with the indicated characteristics. We also assume that  $\infty \notin D$ .

Following (<sup>1</sup>), denote by  $n(w)$  the number of roots of the equation  $T(z) = w$ . It is not difficult to show that the function  $n = n(w)$  is measurable. Therefore one may set

$$P(R) = P(R, D, T) = \frac{1}{2\pi} \int_0^{2\pi} n(Re^{i\varphi}) d\varphi,$$

where the integration is understood in the sense of Lebesgue.

In the plane  $z$  consider the spherical metric  $(r)$ , obtained by stereographic projection of the plane  $z$  onto the Riemann sphere of radius  $r$ , tangent to the plane at the origin.

Then the following holds:

**Theorem 1.** Let  $w = T(z)$  be a  $Q$ -quasiconformal ( $q(z) \leq Q$ ) interior mapping of the domain  $D$  into the plane  $w$ . If  $L_r(R) \equiv L_r(R, D, T)$  is the total length in the spherical metric  $(r)$  of the curves in  $D$  on which  $|T(z)| = R$ , then

$$\int_0^\infty \frac{L_r^2(R) dR}{RP(R)} \leq 2\pi Q S_r(D), \quad (1)$$

where  $S_r(D)$  is the spherical area of the domain  $D$ .

2°. If the domain  $D$  has finite area  $S(D)$  in the ordinary Euclidean metric, then from (1), with the aid of Fatou' s lemma as  $r \rightarrow \infty$ , we obtain

$$\int_0^\infty \frac{L^2(R)}{RP(R)} dR \leq 2\pi Q S(D); \quad (2)$$

here  $L(R) = \lim_{r \rightarrow \infty} L_r(R)$ .

For  $Q = 1$  (the case of an analytic function) we obtain the inequality taken as the basis of the considerations in (1).

For  $Q \geq 1$ , but  $P(R) \equiv 1$ ,  $0 < R < \infty$  (the case of univalent  $Q$ -quasiconformal mappings), inequality (2) has been applied by many authors. Inequality (1) with  $P = 1$  was taken as the basis of our work (4).

3°. **Applications.** Inequalities (1), (2) make it possible to establish a number of facts about inner  $Q$ -quasiconformal mappings quite analogously to the case of analytic functions. We give examples.

- 1) If  $D$  is the disk  $|z| < 1$  and  $T(z) \neq 0$  in  $D$ , then, denoting  $|T(0)| = R_1$ ,  $|T(re^{i\varphi})| = R_2$  for  $0 < r < 1$ ,  $0 \leq \varphi < 2\pi$ , we have

$$\left| \int_{R_1}^{R_2} \frac{dR}{RP(R)} \right| \leq 2Q \left( \ln \frac{1+r}{1-r} + \pi \right). \quad (3)$$

The proof of inequality (3) is obtained from (2) literally as in (1), Theorem 2.2, using at the required place the maximum principle for the modulus of a pseudoharmonic function.

If one additionally requires that  $P(R, D, T) \leq p$ ,  $p = \text{const}$  for  $0 < R < \infty$ , then from (3), for  $0 < r < 1$ , we obtain

$$|T(0)| \left( e^{-\pi} \frac{1-r}{1+r} \right)^{2pQ} < |T(re^{i\varphi})| < |T(0)| \left( e^{\pi} \frac{1+r}{1-r} \right)^{2pQ}. \quad (4)$$

For  $Q = 1$ , inequality (4) gives Cartwright' s result (1, p. 32).

- 2) Let now  $T(z)$  be an “ $p$ -valent on the average with respect to area” inner  $Q$ -quasiconformal mapping of the unit disk  $D$ , i.e.

$$\int_0^R P(\rho, D, T) d\rho^2 \leq pR^2, \quad 0 < R < \infty$$

( $p > 0$  and not necessarily an integer) (see <sup>(1)</sup>, p. 32)). Then, if  $T \neq 0$  in  $D$ ,  $0 < r < 1$ , we get

$$\frac{|T(0)|}{e^{2pQ+1/2}} \left( \frac{1-r}{1+r} \right)^{2pQ} < |T(re^{i\varphi})| < |T(0)| e^{2pQ+1/2} \left( \frac{1+r}{1-r} \right)^{2pQ}. \quad (5)$$

The proof of (5) can be obtained by a literal transfer of the proof from <sup>(1)</sup>, Theorem 2.3 (using (2)).

For  $Q = 1$ , inequality (5) gives Spencer's result.

**Remark 1.** Our results (4) and (5), obtained as elementary consequences of the basic inequality (1), could have been obtained in another way. One may represent our mapping  $T$  in the form of a superposition  $\Phi(f)$  of an analytic function  $\Phi$  and a quasiconformal mapping  $f$  of the disk onto a disk with the same characteristics. After this, one can use the sharp estimate of P. P. Belinskii <sup>(5)</sup>, pp. 328–329):  $|f(z) - z| \leq \rho(r, Q) - r$ ,  $r = |z|$ , and the corresponding result for analytic functions ((4) or (5) when  $Q = 1$ ).

However, in order to obtain estimates in explicit form, it is still necessary here to estimate the quantity  $\rho$  from a certain transcendental equation containing the complete elliptic integral of the first kind  $K(k)$ ,  $k = k(\rho)$ , and its derivative with respect to  $k$ ; moreover, the estimate for  $T$  obtained in this way will not be sharp, since the estimate for analytic functions is not such. In addition, the methods used here will be more subtle (the variational method) than ours.

4°. Let us again consider an arbitrary simply connected domain  $D$ ,  $\infty \in D$ , and two attainable boundary points of  $D$ :  $\zeta_1 = \xi_1 + i\eta$ ,  $\zeta_2 = \xi_2 + i\eta_2$ ,  $\xi_1 < \xi_2$  ( $\xi_1, \xi_2 = \pm\infty$  is not excluded).

The straight lines  $\operatorname{Re} z = x$ ,  $\xi_1 < x < \xi_2$ , intersect  $D$  in a finite or countable set of segments, each of which is a section of  $D$  that divides  $D$  into two subdomains. One of these sections  $\theta_x$  separates  $\xi_1$  and  $\xi_2$  in  $D$ . Let  $\theta(x)$  be the (Euclidean) length of  $\theta_x$ . It is easy to show that the function  $\theta = \theta(x)$  is measurable.

Let  $w = u + iv \equiv T(z)$  be a  $Q$ -quasiconformal mapping of  $D$  onto the strip  $|v| < a/2$ ,  $a > 0$ , such that  $\xi_1, \xi_2$  pass, respectively, into the points  $u = -\infty$ ,  $u = +\infty$ . Put  $\xi_1 < x_1 < x < x_2 < \xi_2$ . Under the indicated mapping,  $\theta_x$  passes into a continuous curve  $l_x$  joining the two boundary straight lines  $v = \pm a/2$ .

If  $u_2(x)$  is the largest, and  $u_1(x)$  the smallest, value of  $u$  on  $l_x$ , then the following holds:

**Theorem 2.** If

$$\int_{x_1}^{x_2} \frac{dx}{\theta(x)} > 2,$$

then

$$u_1(x_2) - u_2(x_1) \geq \frac{1}{Q} \left[ a \int_{x_1}^{x_2} \frac{dx}{\theta(x)} - 4a \right], \quad (6)$$

where the integration is understood in the sense of Lebesgue.

For  $Q = 1$  the theorem gives Ahlfors' result ((<sup>2</sup>, p. 96)); moreover, in (<sup>2</sup>) the boundary of  $D$  is subject to a certain additional regularity condition, which is due to the use in (<sup>2</sup>) of the Riemann integral.

**Remark 2.** All the results of the note are also valid for  $Q$ -quasiconformal mappings in the sense of M. A. Lavrent' ev, and for more general ones—in the sense of I. N. Pesin (<sup>6</sup>).

Tomsk State University  
named after V. V. Kuibyshev

Received  
29 V 1961

## References

- <sup>1</sup> V. K. Kheĭman, *Multivalent Functions*, 1960.
- <sup>2</sup> R. Nevanlinna, *Uniform Analytic Functions*, 1941.
- <sup>3</sup> L. I. Volkovyskii, *Quasiconformal Mappings*, Lviv University Press, 1954.
- <sup>4</sup> G. D. Suvorov, *Siberian Mathematical Journal*, 1, 3, 492 (1960).
- <sup>5</sup> P. P. Belinskii, *Siberian Mathematical Journal*, 1, 3, 303 (1960).
- <sup>6</sup> I. N. Pesin, *Mathematical Collection*, 40 (82), 281 (1956).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*