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Abstract

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ON THE EFFECTIVE SOLUTION OF SINGULAR TRICOMI PROBLEMS FOR THE CHAPLYGIN EQUATION

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In the theory of transonic gas flows an important role is played by the equation of S. A. Chaplygin $\eta z_{\theta\theta} + z_{\eta\eta} + b(\eta) z_\eta = 0$, in which, near the line

$$\eta = 0 \qquad b(\eta) = \sum_{m=0}^{\infty} b_m \eta^m.$$

In the domain $\eta < 0$, in the variables $x = \theta - \frac{2}{3}(-\eta)^{3/2}$, $y = \theta + \frac{2}{3}(-\eta)^{3/2}$, it takes the form

$$G[z] = z_{xy} + A(z_x - z_y) = 0, \tag{1}$$

$$A = \frac{1}{6s} + \sum_{m=0}^{\infty} a_m s^{(2m-1)/3}, \qquad s = y - x; \qquad a_m = \frac{1}{4}(-1)^{m+1}(3/4)^{(2m-1)/3} b_m.$$

We shall call $z(x, y)$ and $\bar{z}(x, y)$ solutions of the first and second singular Tricomi problems if these functions, in a certain domain D $(0 \le x \le y \le x_0)$, satisfy equation (1) and the boundary conditions

$$z(x, x) = \tau(x), \qquad \bar{z}_\eta(x, x) = \nu(x), \qquad z(0, y) = \bar{z}(0, y) = 0, \tag{2}$$

where $\tau(0) = 0$, $\tau(x)$ and $\nu(x) \in C^2[0, x_0]$. Problems of this type occur, for example, in the calculation of the characteristics of a wing profile ($^{1-3}$). To solve problems (2) it is sufficient to find their resolvents of Duhamel type $U(x, y)$ and $\bar{U}(x, y)$—discontinuous integrals of equation (1) with boundary data (see (^{4}), formulas (5)):

$$U(x, x) = \bar{U}_\eta(x, x) = 1, \qquad U(0, y) = \bar{U}(0, y) = 0. \tag{3}$$

For the purpose of constructing $U(x, y)$, make in (1) the substitution $s = y - x$, $t = x/y$, and then, in the equation arising thereby,

$$Q[z] = s(1-t^2)z_{st} - t(1-t)^2 z_{tt} - s^2 z_{ss} + [As(1-t^2) - (1-t)^2]z_t - 2As^2 z_s = 0 \tag{4}$$

put

$$z = U(x,y) = \sum_{n=0}^{\infty} U_n(x,y) = \sum_{n=0}^{\infty} s^{n/3} f_n(t). \tag{5}$$

Then we obtain the recurrent system of ordinary differential equations

$$L_{m+2}[f] = t(1-t)^2 f''_{m+2}(t) - \frac{1}{6}(1-t)[2m-1+(2m+11)t]f'_{m+2}(t) + \frac{1}{9}m(m+2)f_{m+2}(t) = \sum_{n=0}^{m} a_{n/2}\left[(1-t^2)f'_{m-n}(t) - \frac{2}{3}(m-n)f_{m-n}(t)\right]. \tag{6}$$

Here $m = -2, -1, 0, 1, 2, \dots$; $f_{-2}(t) = f_{-1}(t) = 0$; $a_{1/2} = a_{3/2} = a_{5/2} = \dots = 0$. In addition, in order to ensure fulfillment of the boundary conditions (3), we shall assume that $f_0(0) = 0$, $f_0(1) = 1$, and also $f_n(0) = 0$,

$$\lim_{t\to 1}\left[(1-t)^{n/3} f_n(t)\right] = 0, \qquad \text{if } n = 1, 2, \dots \tag{7}$$

For $m = -1, 1, 3, 5, \dots$ the right-hand sides of formulas (7) contain only the functions $f_k(t)$ with odd indices $k = 1, 3, 5, \dots, m$, i.e., these equations have the form $L_{m+2}[f_{m+2}] = \varphi_{m+2}(f_1, f_3, f_5, \dots, f_m)$, where $\varphi_{m+2}(0,0,0,\dots,0) = 0$. In the first of these equations $L_1[f_1] = 0$ $(m = -1)$ the right-hand side $\varphi_1 \equiv 0$, and, as follows from the known properties of Gauss functions, for its solution $f_1(t) = B_1\sqrt{t}\,F({}^4/_3, {}^5/_6, {}^3/_2; t)$, satisfying the requirement $f_1(0) = 0$, the second of conditions (7) can be fulfilled only for $B_1 = 0$, i.e., when $f_1(t) \equiv 0$. This, in turn, means that $\varphi_3 \equiv 0$, and $f_3(t)$ is likewise determined from the homogeneous equation $L_3[f] = 0$. On the other hand, it is not difficult to see that for any values $m = 1, 2, \dots$ the homogeneous equations $L_{m+2}[f] = 0$ cannot yield nontrivial solutions $f^{(0)}_{m+2}(t)$ satisfying both requirements (7) simultaneously. Indeed, in a neighborhood of the point $t = 0$

$$f^{(0)}_{m+2}(t) = (1-t)^{-m/3}\Big[A_{m+2}F({}^5/_6, -m/3, {}^1/_6 - m/3; t) + B_{m+2}t^{5/6+m/3}F({}^5/_3 + m/3, {}^5/_6, {}^{11}/_6 + m/3; t)\Big]. \tag{8}$$

Putting here $A_{m+2} = 0$, we obtain the zero value $f^{(0)}_{m+2}(0) = 0$; however, continuing the remaining hypergeometric function into a neighborhood of the point $t = 1$, we find that the limit

$$\lim_{t\to 1}\left[(1-t)^{(m+2)/3} f^{(0)}_{m+2}(t)\right] = B_{m+2}\Gamma(^2/_3)\Gamma(^{11}/_6 + m/3)\,/\,\Gamma(^5/_6)\Gamma(^5/_3 + m/3)$$

turns to zero only when $B_{m+2} = 0$, i.e., if $f^{(0)}_{m+2}(t) \equiv 0$. Thus, step by step we arrive at the conclusion $f_1(t) \equiv f_3(t) \equiv f_5(t) \equiv \cdots \equiv 0$, and, consequently, the summation in formula (5) must be carried out only over even indices n. In other words, the function U admits an expansion in a series of the form (5) in positive integral powers of the variable $\eta = -(^3/_4 s)^{2/3}$. The integral of the first equation $L_0[f] = 0$ $(m = -2)$ of system (6) with the boundary data $f_0(0) = 0,\ f_0(1) = 1$ is the Joukovsky resolvent $U_0(x, y)$ of the Euler–Poisson equation (4):

$$U_0(x, y) = f_0(t) = B_0 t^{1/6} F(^1/_6,\ ^1/_3,\ ^7/_6; t) = I_t(^1/_6,\ ^2/_3), \tag{9}$$

where $B_0\Gamma(^7/_6)\Gamma(^2/_3) = \Gamma(^5/_6)$. Further, in finding a suitable integral of the second $(m = 0)$ of equations (6),

$$L_2[f] = t(1-t)f_2''(t) + {}^1/_6(1 - 11t)f_2'(t) = a_0(1+t)f_0'(t) \tag{10}$$

one may disregard the solution $f^{(0)}_2(t)$ corresponding to the homogeneous equation $L_2[f] = 0$, since the function needed by us with boundary data (7) is contained among the particular solutions $f^{(1)}_2(t)$ of equation (10) and has the form $f_2(t) = -{}^3/_2 a_0 f_0(t)$. In a similar way, from conditions (6), (7) the functions $f_4(t), f_6(t), \ldots$ are determined uniquely. Formula (9) shows that as t increases from $t = 0$ to $t = 1$, $f_0(t)$ varies within the limits $0 \le f_0(t) \le 1$. Thus, $|f_2(t)| \le {}^3/_2|a_0|$, when $0 \le t \le 1$, and, consequently, in a neighborhood of the line $\eta = 0$ the second term $U_2 = s^{2/3} f_2(t)$ of expansion (5) will be a quantity of at least order $|\eta|$. In an analogous way, estimating $f_n(t)$ for $0 \le t \le 1$, we arrive at the conclusion that near the sonic line $\eta = 0$

$$U = U_0 + O(|\eta|).$$

Comparing the Riemann resolvents U and U_0 makes it possible to give the found connection formulas, which transform the solutions z_0 and $\bar{z}_0$ of the Euler–Poisson equation into the integrals $z, \bar{z}$ of the Chaplygin equation. Such connections, as in (5), are established for the case of transformed initial values $\tau(x) = F(x)\tau_0(x)$, $\nu(x) = P(x)\nu_0(x)$. Let us also note that the well-known maximum principle for solutions $z(x, y)$, as well as the lemma proved in (6), characterizing the behavior of the function $z(x, y)$ at its extremal points, are direct consequences of the corresponding inequalities $0 \le U(x, y) \le 1$, $U_n(x, x) > 0$, $(x, y) \in \bar{D}$, for the resolvent $U(x, y)$.

Let us turn, finally, to the construction of the Riemann kernel $\bar{U}(x, y)$ of the second singular Tricomi problem. For this purpose we shall seek the solution of equation (4) in the form of the series

$$\bar{z} = \bar{U}(x,y) = \sum_{n=2}^{\infty} s^{n/3} f_n(t) = \sum_{n=2}^{\infty} (4/3)^{n/3} (-\eta) n/2\, f_n(t), \tag{11}$$

where, in order to satisfy the conditions (2), we require that $(n = 3, 4, 5, ...)$

$$f_2(0) = 0, \qquad \lim_{t\to 1} \left[{}^3/_2(1-t)f_2'(t) - f_2(t)\right] = ({}^3/_4)^{2/3}, \tag{12}$$

$$f_n(0) = 0, \qquad \lim_{t\to 1}(1-t)^{(n-2)/3}\left[nf_n(t) - 3(1-t)f_n'(t)\right] = 0. \tag{13}$$

Then here too we arrive at the system (6), in which this time $f_0(t) \equiv 0$, so that $f_2(t)$ satisfies the homogeneous equation $L_2[f] = 0$. Its solution with the boundary data (12) is the function $f_2(t) = B_2 t^{5/6} F(5/3, 5/6, 11/6; t)$, where $2B_2\Gamma(1/3)\Gamma(11/6) = (4/3)^{1/3}\Gamma(1/6)$. Further it is not difficult to see that in the present case as well $f_3 = f_5 = f_7 = \cdots = 0$. On the other hand, in computing the function $f_4(t)$ we can no longer confine ourselves to the particular solution $f_4^{(1)}(t) = -{}^3/_2 a_0 f_2(t)$ of the nonhomogeneous equation

$$L_4[f] = t(1-t)^2 f_4''(t) - {}^1/_2(1-t)(1+5t)f_4'(t) + {}^8/_9 f_4(t) = a_0\left[(1-t^2)f_2'(t) - {}^4/_3 f_2(t)\right],$$

so that now $f_4(t)$ must be constructed in the form of the sum $f_4(t) = f_4^{(0)}(t) + f_4^{(1)}(t)$, where $f_4^{(0)}(t)$ is that integral of the equation $L_4[f] = 0$ for which $f_4^{(0)}(0) = 0$, namely

$$f_4^{(0)}(t) = B_4 t^{7/2}(1-t)^{-2/3} F(7/3, 5/6, 5/2; t),$$

and the second of the conditions (13) gives $3^{4/3}\Gamma(5/6)B_4 = -8\sqrt{\pi}\,a_0$. Continuing this process further, in an analogous way we uniquely determine $f_6(t), f_8(t), ...$, and thereby prove the regularity of the expansion (11). In the case of infinitely differentiable initial functions $\tau(x)$ and $\nu(x)$ for the Euler-Poisson equation $(A = 1/6s)$, there hold the symbolic expansions (5):

$$z_0 = B_0 t^{1/6}(1-t)^{2/3}\Xi_1(5/6, 1/6, 1, 7/6; t, -\delta_x)\tau(x),$$

$$\bar{z}_0 = \bar{B}_0 \eta t^{5/6}(1-t)^{-2/3}\Xi_1(1/6, 5/6, 1, 11/6; t, -\delta_x)\nu(x),$$

where Ξ_1 is Humbert' s confluent hypergeometric function, $\delta_x = xD_x$, and $\bar{B}_0 = -(4/3)^{2/3}B_2$. In constructing analogous resolving operators $z = V(x, y, D_x)\tau(x)$ and $z = V(x, y, D_x)\nu(x)$ in the case (1), one may use the following device (4).

By the substitution $z(x,y) = e^{kx}V(x,y)$ we transform (1) to the form

$$G[V] + k(V_y + AV) = 0. \tag{14}$$

Then, in order to find the function $V(x, y, D_x)$, it is enough to determine the solution $V(x, y, k)$ $(k = D_x = \text{const})$ of equation (14) under the boundary conditions $V(x, x) = 1,\ V(0, y) = 0$, i.e. to find the Riemann resolvent of the first singular Tricomi problem for equation (14). The operator $V(x, y, k)$ will be ...

where it should be sought in the form of the series $V(x, y, k) = \sum_{n=0}^{\infty} k^n W_n(x, y)$, substitution of which into (14) gives the system

$$G[W_0] = 0; \qquad G[W_{m+1}] + W_{my} + AW_m = 0 \quad (m = 0, 1, 2, 3, ...). \tag{15}$$

To this system we adjoin the equalities $W_0(0, y) = W_m(0, y) = W_m(x, x) = 0$, $W_0(x, x) = 1$ for $m = 1, 2, 3, ...$. Then it is clear that W_0 coincides with the resolvent $U(x, y)$ computed above. The subsequent functions $W_m(x, y)$, naturally, like W_0, should be sought in the form of expansions $W_m = s^m \sum_{n=0}^{\infty} s^{n/3} \Phi_n^{(m)}(t)$ from the equations $(s = y - x,\ t = x/y)$:

$$Q[W_{m+1}] = st(1-t)W_{mt} - s^2 W_{ms} - As^2 W_m \quad (m = 0, 1, 2, ...). \tag{16}$$

This leads to ordinary differential equations for the functions $\Phi_n^{(m)}(t)$. Thus, for example, in the case when $A = 1/6s$, one may put $W_m = s^m \Phi_m(t)$, and then (16) gives

$$\Delta_{m+1}[\Phi] = t(1-t)^2 \Phi''_{m+1} - (1-t) \cdot [m+1/6 + (m+13/6)t] \Phi'_{m+1} - (m+1/3)(m+1)\Phi_{m+1} = (m+1/6)\Phi_m - t(1-t)\Phi'_m,$$

where $m = -1, 0, 1, 2, ...$, with $\Phi_{-1}(t) \equiv 0$. At the same time the boundary values for W_m are ensured by the equalities $\Phi_0(0) = \Phi_m(0) = 0$, $\Phi_m(1) = 1$, $\lim_{t \to 1}[(1-t)^m \Phi_m(t)] = 0$, if $m = 1, 2, 3, ...$. Hence, first of all, it follows that $\Phi_0(t) = f_0(t)$. Next we establish that for $m = 0$ and $m = 1$ the inhomogeneous equations and the required boundary conditions can be satisfied by setting

$$\Phi_1(t) = \frac{1}{2}\Phi_0(t) - 3t\Phi'_0(t)$$

and

$$\Phi_2(t) = \frac{7}{16}\Phi_1(t) - \frac{3}{8}t\Phi'_1(t).$$

Finally, by induction we find the recurrence formula

$$m(m - 2/3)\Phi_m(t) = (m - 5/6)\Phi_{m-1}(t) - t\Phi'_{m-1}(t) \quad (m = 1, 2, 3, ...),$$

from which, taking into account the known Gauss relations for contiguous hypergeometric functions, we obtain

$$\Phi_m(t) = \mu_m t^{m+1/6} F(m+1/6,\ m+1/3,\ m+7/6;\ t) = \overline{\mu}_m B_t(m+1/6,\ 2/3-m),$$

where $\mu_m(m+1/6) = (-1)^m \overline{\mu}_m$; $m!\Gamma(1/6)\Gamma(2/3)\overline{\mu}_m = \Gamma(5/6)$, and $B_t(p,q)$ is the incomplete Euler beta-function. Analogous constructions are also carried out in constructing the resolving operator $V(x, y, D_x)$.

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Note: Figure translations are in progress. See original paper for figures.