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Abstract

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On the General Form of the Solution of a Homogeneous Differential Equation with Constant Coefficients

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It is well known that the following elementary fact holds: every solution of an ordinary homogeneous differential equation with constant coefficients is a linear combination of certain exponentials (exponential polynomials), in other words, it is the Fourier transform of a linear combination of delta functions (and their derivatives) concentrated at the roots of the corresponding polynomial. The natural formulation of the analogous theorem for equations in partial derivatives is as follows: every solution of the equation

$$p(D)u = 0, \qquad D = (i\partial/\partial x_1, \dots, i\partial/\partial x_n), \tag{1}$$

where $p(s)$ is a polynomial in n variables $s = (s_1, \dots, s_n)$, is the Fourier transform of a functional of finite order, concentrated at the zeros of the polynomial $p(s)$.

We formulate here the main theorem* (Theorem 1), from which it follows, in particular, that an ordinary solution of (1) is represented in the form of the desired integral in the following cases: a) it is an entire analytic function; b) it grows no faster than an exponential of some order; c) it is a derivative of some order of a function of such growth.

In order to be able to speak of the Fourier transform, it is necessary to consider solutions of (1) in one or another space of ordinary or generalized functions. We give the definitions of the functional spaces needed by us.

Following (⁵), by $\mathcal{E}_0^1$ we shall denote the space of all entire analytic functions (of n variables) with the topology of uniform convergence on bounded sets. Further, by $\mathcal{E}_1^0$ (C_1^0) we denote the space of entire (continuous in complex space) functions of at most first order of growth. Topologically, the space $\mathcal{E}_1^0$ (C_1^0) is defined as the inductive limit of the normed spaces of entire (continuous) functions, determined by the norms

$$\|\varphi\|_k = \sup_z |\varphi(z)\exp(-k|z|)|.$$

The spaces S_α^β and $\mathcal{E}_\alpha^\beta$, $\alpha+\beta \geq 1$, are spaces of infinitely differentiable functions which are inductive limits of the normed spaces $S_{\alpha,A}^{\beta,B}$ and $\mathcal{E}_{\alpha,A}^{\beta,B}$, given respectively by the norms

$$\|\varphi\|_{A,B} = \sup_{x,q} \left| \exp(\pm A|x|^{1/\alpha}) \frac{D^q\varphi(x)}{B^q q^{q\beta}} \right|.$$

* While the present paper was being prepared for publication, it became known to the author that Ehrenpreis had announced an analogous theorem.

The spaces S_α^∞ and $\mathcal{E}_\alpha^\infty$, $\alpha > 0$, are also spaces of infinitely differentiable functions, which are inductive limits of normed spaces with norms

$$\|\varphi\|_{A,q} = \sup_{x;\ |i|\leq q} \left|\exp\left(\pm A|x|^{1/\alpha}\right) D^i\varphi(x)\right|.$$

The spaces $\mathcal{E}_0^\beta$ and S_0^β are the spaces of all, respectively of all finite, functions belonging to the Gevrey class of order β. The spaces S_α^β have been studied in detail in (^{1}). Some properties of the spaces $\mathcal{E}_\alpha^\beta$ and the formulas for their Fourier transforms are given in (^{5}).

Let us formulate the main result.

Theorem 1. *Every solution of equation (1) belonging: 1) to the space* $\mathcal{E}_0^1$ *of all entire functions, or 2) to one of the spaces* $S_\alpha^{1-\alpha'}$, $0 < \alpha < 1$, *of generalized functions of infinite order growing at infinity no faster than* $\exp(A|x|^{1/\alpha})$, *is the Fourier transform of some functional of finite order concentrated on the set of roots of the polynomial* $p(s)$.

Thus, among the spaces for which Theorem 1 is valid are all spaces S_α^β $(1 \leq \alpha + \beta \leq \infty)$, where the index α may take any value between zero and one.

We note that, if $\alpha > 1$ $(\alpha = 1)$, then every solution of equation (1) belonging to the space S_α^β, as is not difficult to show, is the Fourier transform of a functional concentrated on the set of all **real** roots of the polynomial $p(s)$ (in any neighborhood of this set). It can be shown that any solution of (1) belonging to the space S_0^β, $\mathcal{E}_0^\beta$ $(1 < \beta \leq \infty)$ can be approximated, in the topology of the dual space, by linear combinations of delta-functions and their derivatives concentrated at the roots of $p(s)$ (see also (^{4})).

The local properties and the behavior at infinity of a solution affect the growth of the functional which is the Fourier transform of this solution. Let us give an exact definition. We shall say that f is a functional of order not higher than q, concentrated at the roots of the polynomial $p(s)$, and growing at infinity no faster than the positive function $F(z)$, if it is a functional on the space of continuous (in the complex space) functions bounded in the norm

$$\|\varphi\| = \sup_{p(z)=0,\ |i|\leq q} \left|F(z)D^i\varphi(z)\right|.$$

Let m be the order of the polynomial $p(s)$. The following result refines Theorem 1.

Theorem 2. *Every solution of equation (1) belonging: 1) to the space* $\mathcal{E}_0^1$*; 2) to one of the spaces* S_α^β, $\mathcal{E}_\alpha^\beta$, $0 < \alpha < 1$ $(1 \leq \alpha + \beta \leq \infty)$, *of generalized, respectively infinitely differentiable functions, growing no faster than some exponential, is the Fourier transform of some functional of order not higher than* m*, concentrated on the set of roots of the polynomial* $p(s)$*. At infinity this functional: 1) decreases no more slowly than an exponential of the first order; 2) grows no faster than* $\exp(-A|\tau|^{1/(1-\alpha)} \pm B|\sigma|^{1/\beta})$, $A > 0$, $B > 0$, *for* $\beta > 0$, *and no faster than* $\exp(-A|\tau|^{1/(1-\alpha)})(|\sigma| + 1)^{\pm M}$, $A > 0$, $M > 0$, *for* $\beta = 0$.

We outline the proofs of Theorems 1 and 2. First consider case 1). From the results of (^{1}), Ch. 3, § 4, it follows that $\mathcal{E}_0^1 = \mathcal{E}_1^{0'}$; consequently, if u is an entire analytic solution of (1), then $\hat{u} \in \mathcal{E}_1^{0'}$.

Let ξ be one of the variables $s_1, \dots, s_n$ entering the polynomial $p(s)$ in degree equal to its order, and let η denote the remaining variables. One can always ensure the existence of such a variable ξ by making a certain real rotation ((^{1}), Ch. 2, §3). Let

$$p(s) = p_0 \prod_{i=1}^{m} [\xi - \xi_i(\eta)].$$

For simplicity we shall assume that, for each η, all the roots $\xi_1(\eta), \dots, \xi_m(\eta)$ are distinct.

We shall need the following simple inequality:

$$|\xi_i(\eta)| \leq c|\eta|, \qquad i = 1, \dots, m. \tag{2}$$

Let H be the operator of taking the interpolation polynomial with respect to the variable ξ at the points $\xi_1(\eta), \dots, \xi_m(\eta)$. By virtue of (2), the operator H is a bounded operator in the space C_1^0. Moreover, the equality

$$Hp\varphi = 0, \qquad \varphi \in C_1^0 \tag{3}$$

is obvious.

The action of the operator H on analytic functions can be represented as follows (^{4}):

$$H\varphi = \frac{1}{2\pi i} \int_\Gamma \frac{p(z,\eta)\varphi(\xi,\eta) - p(\xi,\eta)\varphi(z,\eta)}{(z-\xi)p(z,\eta)}\, dz,$$

where Γ is a contour containing inside it the points $\xi_1(\eta), \dots, \xi_m(\eta)$. From this equality one can see that $H\varphi$ is an analytic function; moreover, H is a bounded operator in $\mathcal{E}_1^0$, and $\varphi - H\varphi = p\psi$, where ψ is some function of the space $\mathcal{E}_1^0$. Hence $(\tilde{u}, \varphi) = (\tilde{u}, H\varphi)$, $\varphi \in \mathcal{E}_1^0$.

Taking this equality as a definition, we extend the functional $\tilde{u}$ to the space $\mathcal{E}_1^0 + pC_1^0 \subset C_1^0$. The resulting functional is continuous in the topology induced from the space C_1^0. This follows from the fact that H is a bounded operator in C_1^0, and from (3): if $\varphi \in \mathcal{E}_1^0 + pC_1^0$, then $H\varphi \in \mathcal{E}_1^0$. And finally, again from (3) it follows that, for any function $\varphi \in C_1^0$,

$$(\tilde{u}, p\varphi) = 0. \tag{4}$$

Extending, by means of the Hahn–Banach theorem, the functional $\tilde{u}$ to the whole space C_1^0, we obtain a functional of order not exceeding m, concentrated, by virtue of (4), on the set of roots of the polynomial $p(s)$. Thus we have established the validity of Theorems 1 and 2 for entire analytic solutions of (1).

We indicate the path of proof of these theorems for the spaces $S_\alpha^{\beta'}$, $0 < \alpha < 1$ $(1 \le \alpha + \beta < \infty)$. In (^{1}) the formula for the Fourier transform $\widetilde{S_\beta^\alpha} = S_\beta^\alpha$ was obtained.

Let all variables $s_1, \dots, s_n$ enter the polynomial $p(s)$ in degree m (this can also be achieved by means of a real rotation); $\eta_1, \dots, \eta_m$ are the corresponding groups of the remaining variables. Let

$$p(s) = p_i \prod_{j=1}^{m} [s_i - s_i^j(\eta_i)]$$

and let $\psi(\xi) \in S_{\beta,B}^{\alpha,A}$ be such a function that $\psi(0) = 1$. We construct the operator $H_i\varphi = H_i\varphi(\xi, \eta_i)$ of taking the interpolation polynomial with respect to the variable s_i from the function $\varphi(s_i, \eta_i)\psi(\xi - s_i)$ at the points $s_i^1(\eta_i), \dots, s_i^m(\eta_i)$.

Next construct a set of entire analytic functions $\chi_i(s)$, $i = 1, \dots, m$, satisfying the following conditions: 1) $|\chi_i(s)| \le c\exp(B|\tau|^{1/(1-\alpha)})$; 2) $|\chi_i(s)| \le c\exp\left(-\frac{1}{A}|\sigma|^{1/(1-\alpha)} + B|\tau|^{1/(1-\alpha)}\right)$ if $|s| \ge 3|s_i|$; 3) $\sum_{i=1}^m \chi_i(s) = 1$. Such a set of functions can be constructed, for example, as follows: first construct continuous functions satisfying these conditions, and then convolve them with any function $\chi_0 \in S_{1-\alpha,B_1}^{\alpha,A_1}$ whose integral over the whole space is equal to 1.

It is not difficult to verify that the operator $H = \sum_{i=1}^m \chi_i(s)H_i$ satisfies the following conditions: 1) H is a bounded operator in $S_{\beta,B_0}^{\alpha,A_0}$ for some A_0, B_0; 2) if $\varphi \in S_{\beta,B_0}^{\alpha,A_0}$, then $\dfrac{\varphi - H\varphi}{p} \in S_\beta^\alpha$; 3) H is a bounded operator in the normed space of continuous functions endowed with the same norm as $S_{\beta,B_0}^{\alpha,A_0}$; 4) if φ belongs to this space, then $Hp\varphi = 0$. Moreover, the smaller the constants A and B, the broader the space $S_{\beta,B_0}^{\alpha,A_0}$ may be taken.

The further arguments are carried out in complete analogy with what was done for the space $\mathcal{E}_0^1$. In the same way the proof of Theorem 2 for the spaces $\mathcal{E}_\alpha^\beta$ follows, thanks to the duality formula

$$\widetilde{\mathcal{E}_\alpha^\beta} = \mathcal{E}_{\beta'}^{\alpha'}, \qquad 0 < \alpha < 1, \quad 1 \leq \alpha + \beta \leq \infty,$$

which follows from the results of [5].

In conclusion we note two classes of equations (1) for which the results obtained can be strengthened. These are, first, elliptic equations, for which Theorems 1 and 2 give the general form of any generalized solution, since, as was proved in [6] (see also [7]), every generalized solution of an elliptic equation is an entire analytic function.

Theorem 3. *If the roots of the polynomial $p(s)$ satisfy the condition*

$$|\operatorname{Im} \xi_i(\eta)| \leq c|\operatorname{Im} \eta| + B,$$

then every solution of (1) belonging to one of the spaces $S_{\beta'}^{\alpha'}$ and $\mathcal{E}_\alpha^\beta$ $(1 < \beta \leq \infty)$ is the Fourier transform of a functional of order not exceeding m, concentrated at the roots of the polynomial $p(s)$, growing at infinity no faster than $\exp(-A|\tau| + B|\sigma|^{1/\beta})$ for $\beta > 0$ and no faster than $\exp(-A|\tau|)(|\sigma| + 1)^{\pm M}$ for $\beta = 0$.

Thus, if the polynomial $p(s)$ satisfies the condition of Theorem 3, then Theorems 1, 2, and 3 give the general form of a solution of (1) belonging to any of the spaces $S_{\alpha'}^{\beta'}$, $\mathcal{E}_\alpha^\beta$ (if S_α^β is nontrivial, see [1]).

The condition of Theorem 3 is satisfied, in particular, by polynomials which are hyperbolic on both sides with respect to the variable ξ (in the sense of [2]) and have positive genus. The proof of Theorem 3 can be obtained by the methods described above.

Theorems 1, 2, and 3, together with the duality formulas for spaces of type S and type $\mathcal{E}$, make it possible to obtain a number of new results for equations with constant coefficients.

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