THEORY OF ELASTICITY

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1961

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Fig. 1

Figure 1: Fig. 1

Abstract

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THEORY OF ELASTICITY

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SOLUTION OF AXISYMMETRIC PROBLEMS OF THE THEORY OF ELASTICITY BY MEANS OF ANALYTIC FUNCTIONS

(Presented by Academician Yu. N. Rabotnov, 20 II 1961)

In papers ([1,2]) the solution of the first and second basic axisymmetric problems of the theory of elasticity, under certain restrictions on the contour of the body, is reduced to determining the boundary values of two analytic functions from two integral equations, obtained in one form for a solid body (Fig. 1a) and in another–for a space with a cavity (Fig. 1b).

In the present paper the integral equations are obtained in a form unified for a solid body and for a space with a cavity. It is shown that the restrictions of paper ([2]) on the contour of the body can be removed. Some solutions of the equations obtained are given.

1. In paper ([2]), by rotating a plane state symmetric with respect to the plane YOZ, the following relations were obtained between the components of this state $(\sigma_{zn}(x,z), \sigma_{xn}(x,z), \ldots)$ and the components of the axisymmetric state of a solid body $(\sigma_{z0}(r,z), \sigma_{r0}(r,z), \ldots)$:

Fig. 1

$$\sigma_{r0} + \sigma_{\theta 0} = \int_{-r}^{r} (\sigma_{xn} + \sigma_{yn}) \frac{dx}{\sqrt{r^2 - x^2}}, \qquad \sigma_{z0} = \int_{-r}^{r} \sigma_{zn} \frac{dx}{\sqrt{r^2 - x^2}},$$

$$\sigma_{r0} - \sigma_{\theta 0} = \int_{-r}^{r} (\sigma_{xn} - \sigma_{yn}) \frac{2x^2 - r^2}{r^2} \frac{dx}{\sqrt{r^2 - x^2}}, \qquad \tau_{rz0} = \int_{-r}^{r} \tau_{xzn} \frac{x\,dx}{r\sqrt{r^2 - x^2}}, \tag{1}$$

$$u_0 = \int_{-r}^{r} u_n \frac{x\,dx}{r\sqrt{r^2 - x^2}}, \qquad w_0 = \int_{-r}^{r} w_n \frac{dx}{\sqrt{r^2 - x^2}}.$$

Here $\sigma_{yn} = \nu(\sigma_{xn} + \sigma_{zn})$, ν is Poisson' s ratio.

Let us introduce into relations (1) representations of the components of the plane state in terms of two analytic functions and perform transformations analogous to those described in paper (^{2}), with the difference that the cut in the plane of the complex variable will be drawn along the body under consideration near its contour, and the integration over the contour will be replaced by integration along the banks of the cut. As a result we obtain:

$$\sigma_{r0}+\sigma_{\theta 0} = -i\int_{t_0}^{-\bar{t}_0} \left[2(1+2\nu)\Phi(t) - (t - t_0 + \bar{t}_0)\Phi'(t) - \Psi(t)\right] \frac{dt}{\sqrt{(t-t_0)(t+\bar{t}_0)}},$$

$$\sigma_{r0} - \sigma_{\theta 0} = -\frac{i}{(t_0+\bar{t}_0)^2}\int_{t_0}^{-\bar{t}_0} \left[2(1-2\nu)\Phi(t) - (t - t_0 + \bar{t}_0)\Phi'(t) + \Psi(t)\right] \times$$

$$\times \frac{2[2t - i(t_0 - \bar{t}_0)]^2 - (t_0+\bar{t}_0)^2}{\sqrt{(t-t_0)(t+\bar{t}_0)}}\, dt,$$

$$\sigma_{z0} = -i\int_{t_0}^{-\bar{t}_0} \left[2\Phi(t) + (t - t_0 + \bar{t}_0)\Phi'(t) + \Psi(t)\right] \frac{dt}{\sqrt{(t-t_0)(t+\bar{t}_0)}}, \qquad (2)$$

$$\tau_{rz0} = -\frac{1}{t_0+\bar{t}_0}\int_{t_0}^{-\bar{t}_0} \left[(t - t_0 + \bar{t}_0)\Phi'(t) + \Psi(t)\right] \frac{2t - t_0 + \bar{t}_0}{\sqrt{(t-t_0)(t+\bar{t}_0)}}\, dt,$$

$$u_0 = -\frac{i}{2\mu(t_0+\bar{t}_0)}\int_{t_0}^{-\bar{t}_0} \left[(3-4\nu)\varphi(t) - \psi(t) - (t - t_0 + \bar{t}_0)\varphi'(t)\right] \times$$

$$\times \frac{2t - t_0 + \bar{t}_0}{\sqrt{(t-t_0)(t+\bar{t}_0)}}\, dt,$$

$$w_0 = -\frac{1}{2\mu}\int_{t_0}^{-\bar{t}_0} \left[(3-4\nu)\varphi(t) + \psi(t) + (t - t_0 + \bar{t}_0)\varphi'(t)\right] \frac{dt}{\sqrt{(t-t_0)(t+\bar{t}_0)}}.$$

Here $t = r + iz$; $t_0 = r_0 + iz_0$; $\bar{t}_0 = r_0 - iz_0$; r, z, r_0, z_0 are the coordinates of points of the contour of the meridional section. The integration is carried out along the positive branch of the root $\sqrt{(t-t_0)(t+\bar{t}_0)}$.

2. In paper (^{2}), with the aid of a linear displacement of the axisymmetric state, the following relations were obtained between the components of this state and the components of a plane state symmetric with respect to the plane YOZ:

$$\sigma_{xn}+\sigma_{yn}=2\int_x^{\infty}(\sigma_{r0}+\sigma_{\theta 0})\frac{r\,dr}{\sqrt{r^2-x^2}},\qquad \sigma_{zn}=2\int_x^{\infty}\sigma_{z0}\frac{r\,dr}{\sqrt{r^2-x^2}},$$

$$\sigma_{xn}-\sigma_{yn}=2\int_x^{\infty}(\sigma_{r0}-\sigma_{\theta 0})\frac{2x^2-r^2}{r\sqrt{r^2-x^2}}\,dr,\qquad \tau_{xzn}=2\int_x^{\infty}\tau_{rz0}\frac{x\,dr}{\sqrt{r^2-x^2}},\tag{3}$$

$$u_n=2\int_x^{\infty}u_0\frac{x\,dr}{\sqrt{r^2-x^2}},\qquad w_n=2\int_x^{\infty}w_0\frac{r\,dr}{\sqrt{r^2-x^2}}.$$

Considering the relations (3) as integral equations, solving them and taking into account the character of the stresses at infinity, we find:

$$\sigma_{r0}+\sigma_{\theta 0}=\frac{1}{\pi}\int_{\infty}^{r}\frac{\partial(\sigma_{xn}+\sigma_{yn})}{\partial x}\frac{dx}{\sqrt{x^2-r^2}},\qquad \sigma_{z0}=\frac{1}{\pi}\int_{\infty}^{r}\frac{\partial\sigma_{zn}}{\partial x}\frac{dx}{\sqrt{x^2-r^2}},$$

$$\sigma_{r0}-\sigma_{\theta 0}=\frac{1}{\pi r^2}\int_{\infty}^{r}\frac{\partial(\sigma_{xn}-\sigma_{yn})}{\partial x}\frac{2x^2-r^2}{\sqrt{x^2-r^2}}\,dx+\frac{C}{r^2};\qquad \tau_{rz0}=\frac{1}{\pi r}\int_{\infty}^{r}\frac{\partial\tau_{xzn}}{\partial x}\frac{x\,dx}{\sqrt{x^2-r^2}},\tag{4}$$

$$u_0=\frac{1}{\pi r}\int_{\infty}^{r}\frac{\partial u_n}{\partial x}\frac{x\,dx}{\sqrt{x^2-r^2}},\qquad w_0=\frac{1}{\pi}\int_{\infty}^{r}\frac{\partial w_n}{\partial x}\frac{dx}{\sqrt{x^2-r^2}}.$$

Here

$$C=-\frac{2}{\pi}\lim_{x\to\infty}[(\sigma_{xp}-\sigma_{yp})x].$$

If the principal vector of the forces applied to the contour is equal to zero, then $C=0$.

We shall regard the derivatives of the components of the plane state with respect to x as the components of another plane state–skew-symmetric with respect to the plane YOZ–which we shall represent by two analytic functions. Introducing these representations into relation (4) and carrying out transformations analogous to those described in Sec. 1, we obtain the values of the components of the axisymmetric state on the contour of the body, expressed in terms of the contour values of two analytic functions. When the principal vector of the forces applied to the contour is zero, these expressions coincide with expressions (2).

Fig. 2

Fig. 2

Figure 2: Fig. 2

3. The circumstance that two different superpositions lead to analogous representations of the components of the axisymmetric state in terms of the contour values of analytic functions compels one to suppose that these representations are sufficiently general, and that one may dispense with the requirement, introduced in paper (2), of single-valuedness of the function $r(z)$ for the half of the contour of the cross section of the body lying on one side of the axis OZ.

We shall show the possibility of superpositions—carried out somewhat differently than in paper (2), but leading to analogous results—even when this restriction is removed. The body, being in a plane deformed state, will be regarded as part of an elastic space loaded by loads Q and P, applied along the contour $r(z)$, and by loads $Q(a, z)$ and $P(a, z)$, distributed outside the meridional section of the body (Fig. 2). (We note that, for given boundary conditions, such loading is multivalued.) By rotating these loads through the angle π about the axis OZ, we obtain an axisymmetric state of the body. As a result of the rotation of the loads of the plane state $Q_1,\ Q_2,\ Q(a, z_0),\ P_1,\ P_2,\ P(a, z_0)$, applied on the straight line $z = z_0$, for $\rho > b_0 > a_0$, there will appear axisymmetric loads of the form (see (1))

$$q(\rho, z_0) = \frac{2Q_1}{\sqrt{\rho^2 - a_0^2}} + \frac{2Q_2}{\sqrt{\rho^2 - b_0^2}} + \int_{a_0}^{b_0} \frac{2Q(a, z_0)}{\sqrt{\rho^2 - a^2}}\, da; \qquad (5)$$

$$p(\rho, z_0) = \frac{2aP_1}{\rho\sqrt{\rho^2 - a_0^2}} + \frac{2b_0 P_2}{\rho\sqrt{\rho^2 - b_0^2}} + \int_{a_0}^{b_0} \frac{2aP(a, z_0)}{\rho\sqrt{\rho^2 - a^2}}\, da.$$

For certain relations between the loads $Q_1,\ Q_2,\ P_1,\ P_2,\ Q(a, z_0)$ and $P(a, z_0)$, $q(\rho, z_0) = p(\rho, z_0) = 0$ for $\rho > b_0 > a_0$. Hence it follows that an axisymmetric state without loads acting inside the meridional section of the body can be obtained even without the restriction imposed on the contour in paper (2).

Let us determine whether, in this case, the dependences (1) can be represented in the form (2). Let the integration of the components of the plane state (see (4)) be carried out along the straight line AF, which on the segments BC and DE passes through the regions where the loads $Q_1,\ Q_2,\ P_1,\ P_2,\ Q(a, z_0),\ P(a, z_0)$ are applied (Fig. 2). These

loads as certain body forces; moreover, Q_1, Q_2, P_1, P_2 may be interpreted as loads $Q_1(a, z_0), Q_2(a, z_0), \ldots$, distributed along the straight line AF over segments of length d so that $Q_1 = Q_1(a, z_0)d,\ Q_2 = Q_2(a, z_0)d, \ldots$ (in what follows we let $d \to 0$). Let us represent the components of the plane state due to the loads $Q = Q(\zeta_0)$, $P = P(\zeta_0)$, applied on the segment DE, in the form of integrals of stresses and displacements at the point ζ, not lying on the straight line AF, due to concentrated forces Q and P applied at the point ζ_0 of the straight line AF. We shall bear in mind that these stresses and displacements are determined by the functions $\Phi(\zeta) = -\dfrac{P+iQ}{8\pi(1-\nu)}\times$

$$\times\frac{1}{\zeta-\zeta_0}, \qquad \Psi(\zeta) = \frac{3-4\nu}{8\pi(1-\nu)}\,\frac{P-iQ}{\zeta-\zeta_0} - \frac{\bar\zeta_0(P+iQ)}{8\pi(1-\nu)}\,\frac{1}{(\zeta-\zeta_0)^2},$$

which are analytic everywhere except for the point $\zeta = \zeta_0$. Bringing the point ζ close to ζ^* on the straight line AF, we use the Sokhotski–Plemelj formulas

$$\lim_{\zeta\to\zeta^*}\frac{1}{2\pi i}\int_a^b \frac{f(\zeta_0)\,d\zeta_0}{\zeta_0-\zeta} = \pm\frac{1}{2}f(\zeta^*) + \frac{1}{2\pi i}\int_a^b \frac{f(\zeta_0)\,d\zeta_0}{\zeta_0-\zeta^*}. \qquad (6)$$

Substituting the plane-state stresses obtained in this way into relations (1), we find the stresses of the axisymmetric state as the sum of two terms corresponding to the first and second terms on the right-hand side of formula (6). If the loads of the plane state are such that, for $\rho > b_0$, $q(\rho, z_0) = p(\rho, z_0) = 0$, then, comparing expressions (1) and (5), one can see that the first terms entering the expressions $\sigma_{z0}, \sigma_{r0} + \sigma_{\theta 0}, \tau_{rz0}$ are equal to zero. The integrals entering the right-hand sides of expressions (6) and appearing here in the sense of the principal value, as well as the integrals entering the expressions for the displacements u and w, are representable by analytic functions and lead to expressions analogous to (2). Similar results can also be obtained for superposition by means of linear displacement of the axisymmetric state.

4. Substituting expressions (2) into the boundary conditions of the problem, we obtain a system of two integral equations for determining the boundary values of two analytic functions. In the case when the bodies under consideration are bounded by one or two concentric spherical surfaces, $\varphi(t), \Phi(t), \ldots$ can be represented by series of the form $\varphi(t) =$

$$= \sum_{k=m}^{n} a_k t^k, \qquad \psi(t) = \sum_{k=m}^{n} b_k t^k.$$

For a solid sphere $m = 0$, $n = \infty$; for a space with a cavity $m = -1$, $n = -\infty$; for a hollow sphere $m = -\infty$, $n = \infty$. From the conditions of symmetry of the plane state with respect to the plane YOZ it follows that the even coefficients a_k and b_k are imaginary, and the odd ones are real. Substituting the series into expressions (2), and using the relation

$$\int_{t_0}^{-\bar t_0} \frac{t^k\,dt}{\sqrt{(t-t_0)(t+\bar t_0)}} = \pi R^k i^{k+1} P_k(\nu)$$

(here $t = Re^{i\alpha}$, R is the radius of the sphere, $\nu = \cos\alpha$, $P_k(\nu)$ is the Legendre polynomial). Multiplying both sides of the equations by $P_s(\nu)\,d\nu$, integrating from -1 to $+1$, and using the orthogonality property of the Legendre polynomials, one can obtain formulas for determining all coefficients of the series.

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Received
16 II 1961

CITED LITERATURE

1. A. Ya. Aleksandrov, DAN, **128**, No. 1 (1959).

2. A. Ya. Aleksandrov, DAN, **129**, No. 4 (1959).

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