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V. A. IL' IN

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Abstract

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V. A. IL' IN

ON THE SYSTEM OF CLASSICAL EIGENFUNCTIONS OF A LINEAR SELF-ADJOINT ELLIPTIC OPERATOR WITH DISCONTINUOUS COEFFICIENTS

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In the present work a number of properties are established for the Green' s functions of the Dirichlet and Neumann problems for a general linear self-adjoint elliptic operator of the second order with discontinuous coefficients. With the aid of these properties the existence of a complete orthonormal system of classical eigenfunctions of the indicated operator is proved. Further, with the aid of the results of the work ([1]), it is proved that the system of classical eigenfunctions of the operator under consideration with discontinuous coefficients coincides with the system of generalized eigenfunctions of the same operator.

1°. Let an open N-dimensional domain g, together with its boundary Γ, lie inside some open domain T. Suppose further that the domain g is divided by some closed surface C lying inside it into two subdomains g_1 (lying inside C) and g_2. Consider in $(g+\Gamma)$ the following eigenfunction problem:

$$L_1 v + \lambda v = \sum_{i,k=1}^{N} \frac{\partial}{\partial x_i}\left[a_{ik}^{(1)}(x)\frac{\partial v}{\partial x_k}\right] - c^{(1)}(x)v + \lambda v = 0 \quad \text{in } g_1,$$

$$L_2 v + \lambda v = \sum_{i,k=1}^{N} \frac{\partial}{\partial x_i}\left[a_{ik}^{(2)}(x)\frac{\partial v}{\partial x_k}\right] - c^{(2)}(x)v + \lambda v = 0 \quad \text{in } g_2, \tag{1}$$

$$v\Big|_{\Gamma} = 0, \qquad [v]\Big|_{C} = 0, \qquad \left[\frac{\partial v}{\partial \nu}\right]\Bigg|_{C} = 0,$$

where

$$[v] = v\Big|_{C-0} - v\Big|_{C+0}; \qquad \left[\frac{\partial v}{\partial \nu}\right]\Bigg|_{C} = \frac{\partial v}{\partial \nu_1}\Bigg|_{C-0} - \frac{\partial v}{\partial \nu_2}\Bigg|_{C+0};$$

ν_1 is the conormal, external with respect to the domain g_1, for the operator L_1; ν_2 is the conormal, internal with respect to the domain g_2, for the operator L_2;

and the symbols $C-0$ and $C+0$ mean that limiting values are taken respectively from the inner side and from the outer side of the surface C with respect to g_1.

Let the following four conditions be fulfilled: 1) the surface C belongs to the Lyapunov class, and the surface Γ is regular*; 2) the functions $a_{ik}^{(m)}(x)$ and $c^{(m)}(x)$ $(m=1,2)$ are defined and belong to the classes**: $a_{ik}^{(1)}(x) \in C^{(1,\mu)}$ in the domain (g_1+C), $a_{ik}^{(2)} \in C^{(1,\mu)}$ in $(T-g_1)$, $c^{(1)}(x) \in C^{(0,\mu)}$ in (g_1+C), $c^{(2)}(x) \in C^{(0,\mu)}$ in $(T-g_1)$; 3) the coefficients $a_{ik}^{(m)}(x)$ $(m=1;2)$ everywhere in the domains of their definition satisfy the conditions of ellipticity; 4) $c^{(m)}(x) \geq 0$ everywhere in the domains of their definition.

We shall call these four conditions **conditions A**.

* A surface is called regular if, in the domain bounded by this surface, the Dirichlet problem for the Laplace equation is solvable for every continuous boundary function.

** Definitions of all the classes used in this article may be found in ([2]).

First of all, the properties of the Green's function are established for the Dirichlet problem with discontinuous coefficients of the form

$$L_1 u = -f_1(x) \text{ in } g_1, \qquad L_2 u = -f_2(x) \text{ in } g_2,$$
$$u\big|_{\Gamma} = 0, \qquad [u]\big|_C = 0, \qquad \left[\frac{\partial u}{\partial \nu}\right]\bigg|_C = 0. \tag{2}$$

Theorem 1. Suppose that conditions A are satisfied. Then there exists a Green' s function $K(x,y)$ of problem (2), and this function satisfies the following requirements: 1) $K(x,y)$ is symmetric with respect to x and y; 2) $K(x,y) \in N^{(2)}$ for $x, y \in (g+\Gamma)$ (i.e., $K(x,y)$ is continuous jointly in x, y everywhere in the closed domain $(g+\Gamma)$, except at the points $x=y$, and for it the uniform estimate in the closed domain $(g+\Gamma)$,

$$K(x,y) = O\left(r_{xy}^{2-N}\right),$$

is valid*); 3) if $f_1(x) \in C^{(0,\mu)}$ in g_1 and $C^{(0)}$ in (g_1+C), and $f_2(x) \in C^{(0,\mu)}$ in g_2 and $C^{(0)}$ in $(g_2+C+\Gamma)$, then the unique classical solution $v(x)$ of problem (2) is determined by the formula

$$v(x) = \int_{g_1} K(x,y) f_1(y)\, dy + \int_{g_2} K(x,y) f_2(y)\, dy. \tag{3}$$

Let us outline the proof of Theorem 1. We shall seek the Green' s function $K(x,y)$ of problem (2) in the form

$$K(x,y) = K_1(x,y) + K_2(x,y) + w(x,y), \tag{4}$$

where $K_1(x,y)$ is the Green' s function of the operator L_1 in the domain g_1, extended identically by zero to the domain g_2; $K_2(x,y)$ is the Green' s function of the operator L_2 in the domain g_2, extended identically by zero to the domain g_1, and $w(x,y)$, for any fixed y belonging to g_1 or g_2, is a solution of the following Dirichlet problem with discontinuous coefficients:

$$L_1 w = 0 \text{ in } g_1, \qquad L_2 w = 0 \text{ in } g_2, \tag{5}$$
$$w\big|_\Gamma = 0, \qquad [w]\big|_C = 0, \qquad \left[\frac{\partial w}{\partial \nu}\right]\bigg|_C = \theta(x,y),$$

where $\theta(x,y) = -\partial K_1(x,y)/\partial\nu_{1x}$ for $y \in g_1$, and $\theta(x,y) = \partial K_2(x,y)/\partial\nu_{2x}$ for $y \in g_2$.

By virtue of the properties of the Green' s functions $K_1(x,y)$ and $K_2(x,y)$ and of one result of Giraud ([3]), $\theta(x,y)$, as a function of x, for any fixed $y \in (g_1 + g_2)$, belongs on the surface C to the class $C^{(0,\mu)}$. But then, by virtue of the principal result of work ([1]), the solution of problem (5) exists and is determined by the formula

$$w(x,y) = \int_C K_0(x,t)\,\mu(t,y)\,ds_t, \tag{6}$$

where $K_0(x,y)$ is the kernel constructed in work ([1]), and $\mu(x)$ is the solution of the integral equation

$$\mu(x,y) = \int_C F(x,t)\mu(t,y)\,ds_t + \theta_1(x,y), \tag{7}$$

whose kernel $F(x,y) \in N^{(1+\delta,\gamma)}$ for $x,y \in C$, $\delta > 0$, $\gamma < \delta$.

By virtue of one delicate result from the theory of integral equations (see ([2]), p. 59), the resolvent $\Gamma(x,y)$ of the kernel $F(x,y)$ in any case belongs to $\overline{N}^{(1+\delta)}$ for $x,y \in C$. But then, expressing the solution of equation (7) through the resolvent

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$$* \text{ Here } N \geq 3. \text{ For } N = 2, \quad K(x,y) = O\left(\ln\frac{2D}{r_{xy}}\right),$$

where D is the diameter of the domain g.

and, substituting this expression into (6), we obtain for $w(x,y)$ the representation

$$w(x, y) = \int_C M(x, t)\, \theta(t, y)\, ds_t, \tag{8}$$

where by $M(x, y)$ is denoted a kernel belonging to the class $N^{(2)}$ for $x \in (g + \Gamma)$, $y \in C$. With the aid of representation (8), taking into account the specific form of the function $\theta(t, y)$, it is not difficult to justify requirement 2) of Theorem 1.

For the proof of property 1), the fact of the coincidence of the classical and generalized solutions of the Dirichlet problem (2), established in item 2° of paper ([1]), is used in an essential way.

Finally, property 3) is proved by relying on the symmetry of $K(x, y)$, the fact of the coincidence of the classical and generalized solutions of problem (2), and Green' s formula.

2°. We pass to the question of the existence of classical eigenfunctions of problem (1).

Definition. A **classical eigenfunction** of problem (1) is a function $v(x)$ satisfying the following requirements: 1) $v(x) \not\equiv 0$; 2) $v(x)$ belongs to the class $C^{(0)}$ in the closed domain $(g + \Gamma)$, to the class $C^{(1)}$ in each of the domains $(g_1 + C)$ and $(g_2 + C)$, and to the class $C^{(2)}$ in each of the open domains g_1 and g_2; 3) $v(x)$, for some λ, satisfies in the usual classical sense all the conditions of problem (1).

Theorem 2. *If conditions A are satisfied, then there exists a complete (in $\mathcal{L}_2$) orthonormal system of classical eigenfunctions of problem (1).*

For the proof of Theorem 2, one investigates the homogeneous integral equation

$$v(x) = \lambda \int_g K(x, y)\, v(y)\, dy, \tag{9}$$

whose kernel is the Green' s function of problem (2) constructed above. The properties of the Green' s function established in Theorem 1 make it possible to prove that this equation has a countable system of eigenfunctions $\{v_n(x)\}$, that each of these functions is simultaneously a classical eigenfunction of problem (2), and that the system of eigenfunctions is complete in $\mathcal{L}_2(g)$.

3°. Using the result of item 2° of paper ([1]) and the scheme set out in paper ([4]), it is easy to prove the following assertion:

Theorem 3. *The complete system of classical eigenfunctions of problem (1) coincides with the complete system of generalized * eigenfunctions of this problem.*

Remark 1. Theorems analogous to Theorems 1, 2, and 3 also hold for the eigenfunctions of the second and third boundary-value problems, i.e. for the eigenfunctions of problem (1) in which the condition $v|_\Gamma = 0$ is replaced by $(\partial v/\partial \nu_2 + hv)|_\Gamma = 0$, where $h(x) \geqslant 0$.

In this case, to the definition of an eigenfunction one must add $v(x) \in C^{(1)}$ in $(g_2 + C + \Gamma)$, and to conditions A one must make two additions: 1) Γ is a Lyapunov surface; 2) $h(x)$ is continuous and nonnegative on Γ.

4°. **Theorem 4.** *If conditions A are satisfied, then there exists a constant* C_0 *such that uniformly in the closed domain* $(g + \Gamma)$ *the inequality*

$$|v_n(x)| < C_0 \lambda_n^{N/4}. \tag{10}$$

holds. Here $v_n(x)$ *denotes the eigenfunction of problem (1), normalized in* $\mathcal{L}_2(g)$, *corresponding to the eigenvalue* λ_n.

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* For the definition of generalized eigenfunctions see, for example, ([4]).

For the proof of Theorem 4 one should use Theorem 3 and apply the method developed in [5].

Remark 2. All the results of the present paper carry over to the case where inside the surface Γ there lie n closed surfaces of discontinuity of the coefficients, $C_1, C_2, \dots, C_n$, belonging to the Lyapunov class, and on each of these surfaces the boundary conditions $[v]_{C_i} = 0$, $[\partial v/\partial \nu]_{C_i} = 0$ $(i = 1, 2, \dots, n)$ are prescribed. In this case some of the surfaces C_i may lie inside others.

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Moscow State University
named after M. V. Lomonosov

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REFERENCES

[1] V. A. Il' in, DAN, 137, No. 1 (1961).
[2] K. Miranda, *Equations with partial derivatives of elliptic type*, IL, 1957.
[3] G. Giraud, Bull. Soc. Math. de France, 61, 1 (1933).
[4] V. A. Il' in, I. A. Shishmarev, Izv. AN SSSR, ser. matem., 24, No. 5 (1960).
[5] V. A. Il' in, I. A. Shishmarev, Izv. AN SSSR, ser. matem., 24, No. 6 (1960).

Note: Figure translations are in progress. See original paper for figures.