

INTEGRAL TRANSFORMATIONS ASSOCIATED WITH COMPLEXES OF LINES IN COMPLEX AFFINE SPACE

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Abstract

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MATHEMATICS

Consider the three-dimensional complex affine space of points $z = (z^1, z^2, z^3)$ and in it all possible complex lines. We shall specify these lines by Plücker coordinates $\alpha = (\alpha^1, \alpha^2, \alpha^3)$ and $p = (p_1, p_2, p_3)$. By definition, α is the direction vector of the line, and p is the vector product $p = [\alpha, z]$ of the vector α and the radius vector of an arbitrary point z of the line*.

The Plücker coordinates are related by the relation $\alpha^1 p_1 + \alpha^2 p_2 + \alpha^3 p_3 = 0$. Any homogeneous equation $F(\alpha, p) = 0$ among the coordinates α, p , not following from the equation $\alpha^1 p_1 + \alpha^2 p_2 + \alpha^3 p_3 = 0$, determines a certain three-parameter family of lines, called a **complex** ⁽¹⁾. In what follows it is assumed that F is a homogeneous polynomial in α and p .

Let us formulate the problem of integral geometry for a complex of lines. Suppose a function $f(z) = f(z^1, z^2, z^3)$ is given. We shall assume that it is differentiable a sufficient number of times (with respect to z and \bar{z}) and decreases sufficiently rapidly at infinity. Define the integrals of the function $f(z)$ over the lines α, p of the complex as follows:

$$\varphi(\alpha, p) = \frac{i}{2} \int f(\alpha t + \beta) dt d\bar{t}, \quad (1)$$

where β is an arbitrary point of the line α, p , i.e. $p = [\alpha, \beta]**$. The problem consists in reconstructing the original function $f(z)$ from the function $\varphi(\alpha, p)$, i.e. in obtaining an inversion of formula (1).

In the present note this problem is solved for complexes satisfying the following additional condition.

Tangency condition. Let α, p be an arbitrary line of the complex; let z', z'' be arbitrary points on the line α, p . Consider all lines of the complex passing through z' or z'' . These lines form two conical surfaces with common generator $\alpha, p***$. It is required that these conical surfaces always be tangent to each other along the common generator α, p .

The tangency condition is satisfied by complexes of the following form: the complex of all lines tangent to a certain (algebraic) surface; the complex of all

lines intersecting a certain (algebraic) curve; the complex of all lines parallel to the generators of a certain (algebraic) conical surface.

* Thus, the equality $p = [\alpha, z]$ may be regarded as the vector equation of a line given by Plücker coordinates α and p .

** By definition, $\frac{i}{2} dt d\bar{t} = ds dt$, where $t = \sigma + i\tau$.

*** Not counting the special case when all lines passing through one of the points z', z'' belong to the complex. In this case we shall assume the tangency condition to be fulfilled.

For a complex given by the equation $F(\alpha, p) = 0$ to satisfy the tangency condition, it is necessary and sufficient that the Plücker coordinates of the lines of the complex satisfy the relation

$$F'_{\alpha^1} F'_{p_1} + F'_{\alpha^2} F'_{p_2} + F'_{\alpha^3} F'_{p_3} = 0.$$

For complexes satisfying the tangency condition, the problem of integral geometry has a unique solution. This solution is local. Namely, in order to reconstruct the value of the function f at a certain point z_0 , it suffices to know only the integrals over the lines of the complex passing through the point z_0 , and over the lines of the complex infinitely close to them (moreover, it is possible to consider only infinitely close **parallel** lines). It turns out that the complexes under consideration are, in a certain sense, the most general complexes for which the solution of the problem of integral geometry is unique and local (see the remark below). We shall give the solution of the problem.

Let the complex be given by the equation $F(\alpha, p) = 0$. We shall seek the value of the function $f(z)$ at a certain point z_0 . To this end, first draw through the point z_0 all possible lines of the complex. The direction vectors α of these lines satisfy the equation

$$G(\alpha; z_0) \equiv F(\alpha, [\alpha, z_0]) = 0. \quad (2)$$

Introduce the notation $a_i(\alpha; z_0) = G'_{\alpha^i}(\alpha; z_0)$, $i = 1, 2, 3$. Then shift each of the lines α, p of the complex passing through the point z_0 , parallel to itself, by an infinitesimally small distance. The Plücker coordinates α do not change under this shift, while the coordinates p acquire the increment $dp = (dp_1, dp_2, dp_3)$. It can be shown that, in a complex satisfying the tangency condition, the vector (dp_1, dp_2, dp_3) is proportional to the vector (a_1, a_2, a_3) .

Introduce the operators of "infinitesimal parallel displacement of the lines of the complex passing through the point z_0 " :

$$L_{z_0} \varphi = \sum a_i(\alpha; z_0) \varphi'_{p_i}(\alpha, [\alpha, z_0]),$$

$$\bar{L}_{z_0} \varphi = \sum \overline{a_i(\alpha; z_0)} \varphi'_{\bar{p}_i}(\alpha, [\alpha, z_0]),$$

where α satisfies equation (2).

Theorem. *Suppose that the complex $F(\alpha, p) = 0$ satisfies the tangency condition, and suppose that in the cone formed by the lines of the complex passing through the point z_0 , the point z_0 is the only singular point. Then the value of the function $f(z)$ at the point z_0 is expressed in terms of the integrals $\varphi(\alpha, p)$ of the function $f(z)$ over the lines of the complex by the following inversion formula:*

$$f(z_0) = C_{z_0} \int_{\Gamma} L_{z_0} \bar{L}_{z_0} \varphi(\alpha, [\alpha, z_0]) \omega_{z_0}(\alpha) \overline{\omega_{z_0}(\alpha)}. \quad (3)$$

Integration is carried out over an arbitrary contour Γ on the cone (2), intersecting each generator of the cone once; $\omega_{z_0}(\alpha)$ is a differential form on the cone (2) of the form

$$\omega_{z_0}(\alpha) = a_3^{-1}(a_1 d\alpha^2 - a_2 d\alpha^1) = a_1^{-1}(a_2 d\alpha^3 - a_3 d\alpha^2) = a_2^{-1}(a_3 d\alpha^1 - a_1 d\alpha^3).$$

The constant C_{z_0} in formula (3) is expressed by the formula

$$C_{z_0}^{-1} = \pi \Delta \int_{\Gamma} B(a_1, a_2, a_3) A^{-2}(\alpha^1, \alpha^2, \alpha^3) \omega_{z_0}(\alpha) \overline{\omega_{z_0}(\alpha)},$$

where A, B are an arbitrary pair of conjugate Hermitian positive-definite forms, and Δ is the discriminant of the form A .

* In fact, no such contour Γ intersecting each generator of the cone once exists, and the preceding expression should be understood as follows. The space of generators of the cone is divided into sufficiently small regions, and for each region a contour Γ_i is taken that intersects the generators of this region once. Integral (3) is defined as the sum of the integrals over Γ_i . It is easy to show that the integrals over Γ_i do not depend on the choice of the contours Γ_i , and their sum does not depend on the manner in which the space of generators is divided into parts.

Earlier the inversion formula was obtained by another method for special cases, namely in (2) for the complex of lines intersecting a circle, and then in (3) for the complex of lines intersecting an arbitrary algebraic curve.

Remark. The inversion formula can also be obtained in a more general form:

$$f(z_0) = C_{z_0} \int_{\Gamma} (\mathcal{L}_{z_0} + w)(\bar{\mathcal{L}}_{z_0} + \bar{w}) \varphi(\alpha, [\alpha, z_0]) \omega_{z_0}(\alpha) \overline{\omega_{z_0}(\alpha)}, \quad (4)$$

where $\mathcal{L}_{z_0}, \overline{\mathcal{L}}_{z_0}$ are operators of an infinitesimal (not necessarily parallel) displacement of the lines of the complex passing through the point z_0 :

$$\mathcal{L}_{z_0}\varphi = \sum (u^i \varphi'_{\alpha^i} + v_i \varphi'_{p_i}),$$

$$\overline{\mathcal{L}}_{z_0}\varphi = \sum (\bar{u}^i \varphi'_{\bar{\alpha}^i} + \bar{v}_i \varphi'_{\bar{p}_i}),$$

$w = w(\alpha)$ is some function. It is not difficult to find necessary and sufficient conditions on the functions u^i, v_i, w under which the inversion formula (4) holds. We shall not give these conditions here.

An inversion formula of the form (4) holds only for those complexes that satisfy the tangency condition.

There exist complexes of lines for which the problem of integral geometry has a non-unique solution. As an example, let us indicate the first-order complex $k_1\alpha^1 + k_2\alpha^2 + k_3\alpha^3 + l^1p_1 + l^2p_2 + l^3p_3 = 0$, where $k_1l^1 + k_2l^2 + k_3l^3 \neq 0^*$.

For a first-order complex let us pose another problem of integral geometry, equivalent to the preceding one: knowing the integrals of the function $f(z)$ over the lines of the complex, compute the integral of the function $f(z)$ over an arbitrary line.

With a first-order complex there is associated an involutive transformation in the space of all lines that preserves the lines of the complex. Namely, if through the lines of the complex that intersect a given line π , then these lines also intersect some other line π' . We shall call the line π' *conjugate* to the line π with respect to the complex. It turns out that, *if the integrals of the function $f(z)$ over the lines of the complex $k_1\alpha^1 + k_2\alpha^2 + k_3\alpha^3 + l^1p_1 + l^2p_2 + l^3p_3 = 0$, where $k_1l^1 + k_2l^2 + k_3l^3 \neq 0$, are known, then only the sum of the integrals of the function $f(z)$ over any pair of conjugate lines is uniquely determined. Thus the problem of integral geometry has a unique solution in the class of those functions $f(z)$ for which their integrals over conjugate lines coincide.*

For simplicity consider the complex $\alpha^1 = \lambda p_1$, where $\lambda \neq 0$ (in view of $\alpha^1 = \lambda p_1$ any first-order complex can be reduced by a suitable affine transformation of the space to this equation). The line conjugate to the line $(\alpha^1, \alpha^2, \alpha^3; p_1, p_2, p_3)$ with respect to the complex $\alpha^1 = \lambda p_1$ has Plücker coordinates $(\lambda p_1, \alpha^2, \alpha^3; \lambda^{-1}\alpha^1, p_2, p_3)$. *The sum of the integrals of the function $f(z)$ over a pair of lines conjugate with respect to the complex $\alpha^1 = \lambda p_1$ is expressed in terms of the integrals of the function $f(z)$ over the lines of the complex as follows:*

$$\varphi(\alpha_0^1, \alpha_0^2, \alpha_0^3; p_1^0, p_2^0, p_3^0) + \varphi(\lambda p_1^0, \alpha_0^2, \alpha_0^3; \lambda^{-1}\alpha_0^1, p_2^0, p_3^0) =$$

$$= C|\alpha_0^1 - \lambda p_1^0|^2 \int \overline{L}L\varphi(\lambda, \alpha^2, \alpha^3; 1, p_2, p_3) |\alpha_0^2 \alpha^3 - \alpha_0^3 \alpha^2|^{-4} d\alpha^2 d\alpha^3 d\bar{\alpha}^2 d\bar{\alpha}^3. \quad (5)$$

* The geometry of first-order complexes is considered in detail in ⁽¹⁾.

Here $L\varphi = \alpha^2 \varphi_{p_3} - \alpha^3 \varphi_{p_2}$, $\overline{L}\varphi = \bar{\alpha}^2 \varphi_{\bar{p}_3} - \bar{\alpha}^3 \varphi_{\bar{p}_2}$ are operators of an infinitesimal parallel shift in the complex; the integral is taken over the set of all lines of the complex intersecting the given pair of conjugate lines

$$(\alpha_0^1, \alpha_0^2, \alpha_0^3, p_1^0, p_2^0, p_3^0) \quad \text{and} \quad (\lambda p_1^0, \alpha_0^2, \alpha_0^3, \lambda^{-1} \alpha_0^1, p_2^0, p_3^0)^*.$$

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REFERENCES

- ¹ F. Klein, *Higher Geometry*, 1939.
- ² I. M. Gelfand, *Uspekhi Mat. Nauk*, **15**, no. 2 (92), 155 (1960).
- ³ A. A. Kirillov, *Dokl. Akad. Nauk SSSR*, **137**, no. 2 (1961).

* When $\alpha_0^2 = \alpha_0^3 = 0$, the integral in (5) should be understood in the sense of the limiting value as $\alpha_0^2, \alpha_0^3 \rightarrow 0$. Note that when $\alpha_0^2 = \alpha_0^3 = 0$, one of the conjugate lines becomes infinitely distant, and therefore the integral of φ over this line vanishes.

Note: Figure translations are in progress. See original paper for figures.

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