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Abstract

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SOLUTION OF THE CAUCHY PROBLEM FOR HYPERBOLIC EQUATIONS BY THE METHOD OF FINITE DIFFERENCES

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1. In this paper the stability of certain difference schemes for hyperbolic equations of higher orders with an arbitrary number of independent variables is proved, and a proof of the existence of a solution of the Cauchy problem by the method of finite differences is given. We consider an explicit difference scheme, convergent for a sufficiently small ratio of the steps in the time and space coordinates, and two implicit schemes, convergent for any bounded ratio of the steps.

The essential point in the paper is the construction of a difference separating operator, which makes it possible to obtain energy inequalities for solutions of difference equations corresponding to the Cauchy problem for a hyperbolic equation. The separating operator was used earlier in the works of Leray (^{1}) and Gårding (^{2}) for deriving a priori estimates of solutions of the Cauchy problem for hyperbolic equations of higher orders. A solution of the Cauchy problem for hyperbolic systems by the method of finite differences was obtained in (^{3}) with the aid of an implicit scheme, the derivation of the energy estimate being carried out as in the work of I. G. Petrovsky (^{4}). Difference schemes for some classes of hyperbolic systems were also considered in (^{5,6}), etc.

2. **Notation and definitions.** Let $\{x\} = \{x_0, x'\} = \{x_0, x_1, \dots, x_n\}$ be an $(n+1)$-dimensional space; let Ω be a domain in the n-dimensional space $\{x'\} = \{x_1, x_2, \dots, x_n\}$, defined by the inequalities $0 \le x_i \le 2\pi$ $(i = 1, 2, \dots, n)$; $Q_T = \{0 < x_0 \le T,\ x' \in \Omega\}$ be a domain in the space $\{x\}$. Let $D_i \equiv \dfrac{\partial}{\partial x_i}$ $(i = 0, 1, \dots, n)$; $\alpha = \{\alpha_0, \alpha'\} = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$;

$$|\alpha| = \sum_{i=0}^{n} \alpha_i;$$

α_i are nonnegative integers;

$$D^\alpha \equiv D_0^{\alpha_0} D_1^{\alpha_1} \cdots D_n^{\alpha_n}.$$

All functions under consideration are assumed to be periodic in x' with period 2π. In Q_T we seek a solution, periodic in x', of the equation

$$a(x, D)u \equiv \sum_{|\alpha| \leq m+1} a_\alpha(x) D^\alpha u = f, \tag{1}$$

satisfying the conditions

$$D_0^i u\big|_{x_0=0,\ x' \in \Omega} = \varphi_i(x'), \qquad i = 0, 1, \dots, m, \tag{2}$$

where $a(x, D)$ is a normal hyperbolic operator in Q_T, i.e. $a_{m+1,0,0,\dots,0}(x) \equiv 1$, and the equation $a_0(x, \lambda, \xi_1, \dots, \xi_n) = 0$ has, for all real $\xi' = \{\xi_1, \dots, \xi_n\}$ satisfying the condition $\sum_{i=1}^n \xi_i^2 \neq 0$, and all $x \in Q_T$, $m+1$ distinct real roots $\lambda_1(x, \xi'), \dots, \lambda_{m+1}(x, \xi')$, where

$$a_0(x, D) \equiv \sum_{|\alpha| = m+1} a_\alpha(x) D^\alpha$$

is the principal part of the operator $a(x, D)$. We shall assume that the coefficients $a_\alpha(x)$ satisfyo

satisfy in Q_T the Lipschitz condition with respect to $x_0, x_1, \dots, x_n$ for $|\alpha| = m+1$ and the Lipschitz condition with respect to $x_1, x_2, \dots, x_n$ for $|\alpha| \leqslant m$. The separating operator (see (2)) is the operator $b(x, D) = \dfrac{1}{m+1} \dfrac{\partial}{\partial D_0} a_0(x, D)$.

In the space $\{x\}$ construct a grid with steps Δx_0, $\Delta x_i = \Delta x$ $(i = 1, 2, \dots, n)$ such that $T = M_\Delta \Delta x_0$, $2\pi = N_\Delta \Delta x$; M_Δ, N_Δ are natural numbers. Let $\varkappa = \Delta x_0 / \Delta x$. The set of points $x' = k' \Delta x \in \{x'\}$, where $k' = \{k_1, \dots, k_n\}$ are integers satisfying the inequalities $0 \leqslant k_i \leqslant N_\Delta - 1$, will be denoted by Ω_Δ. Let u be a function defined at the nodes of the grid. Then by u_Δ we denote the function defined by the equalities $u_\Delta(x) = u(k_0 \Delta x_0, k_1 \Delta x, \dots, k_n \Delta x)$ for $k_0 \Delta x_0 \leqslant x_0 < (k_0 + 1)\Delta x_0$, $k_i \Delta x \leqslant x_i < (k_i + 1)\Delta x$ $(i = 1, 2, \dots, n)$.

Next denote

$$\overset{\pm i}{u} = u(x_0, \dots, x_i \pm \Delta x_i, \dots, x_n), \qquad J_1 u = \frac{1}{2}(\overset{+0}{u} + u),$$

$$J_2 u = \frac{1}{2}(\overset{+0}{u} + \overset{-0}{u}), \qquad u_{x_i} = \frac{1}{\Delta x_i}(\overset{+i}{u} - u), \qquad u_{\bar{x}_i} = \frac{1}{\Delta x_i}(u - \overset{-i}{u}), \qquad \Delta_i u = \frac{1}{2}(u_{x_i} + u_{\bar{x}_i})$$

$$(i = 0, 1, \dots, n), \qquad \Delta^\alpha = \Delta_0^{\alpha_0} \Delta_1^{\alpha_1} \cdots \Delta_n^{\alpha_n}, \quad \overline{\Delta}_0 u = u_{x_0}, \quad \overline{\Delta}_i u = J_1(\Delta_i u) \quad (i = 1, 2, \dots, n),$$

$$\overline{\Delta}^\alpha = \overline{\Delta}_0^{\alpha_0} \overline{\Delta}_1^{\alpha_1} \cdots \overline{\Delta}_n^{\alpha_n}, \quad \widetilde{\Delta}_0 u = \Delta_0 u, \quad \widetilde{\Delta}_i u = J_2(\Delta_i u) \quad (i = 1, 2, \dots, n),$$

$$\widetilde{\Delta}^{\alpha} = \widetilde{\Delta}_0^{\alpha_0}\widetilde{\Delta}_1^{\alpha_1}\cdots\widetilde{\Delta}_n^{\alpha_n}.$$

Let C^∞ be the set of infinitely differentiable functions in the strip $0 \leqslant x_0 \leqslant T$, periodic in x' with period 2π; $H^{k,j}(Q_T)$ is the Hilbert space obtained by completing C^∞ in the norm

$$\left(\int_{Q_T}\sum_{\substack{|\alpha|\leqslant k+j\\ \alpha_0\leqslant k}}|D^\alpha u|^2\,dx\right)^{1/2}, \qquad H^{k,0}\equiv H^k.$$

3. Stable difference schemes for problem (1), (2)

In this section we shall assume that φ_i and f are defined at every point $x \in \overline{Q}_T$.

Scheme I. Define the function u as follows. Put

$$u(k_0\Delta x_0, k_1\Delta x, \dots, k_n\Delta x) = \sum_{i=0}^{m}\varphi_i(k_1\Delta x,\dots,k_n\Delta x)\frac{(k_0\Delta x_0)^i}{i!} \tag{3}$$

for $0 \leqslant k_0 \leqslant 2m+1$ and arbitrary numbers $k_1, k_2, \dots, k_n$. For points $x = \{k_0\Delta x_0, k_1\Delta x, \dots, k_n\Delta x\}$ with $m+1 \leqslant k_0 \leqslant M_\Delta - 1$, $0 \leqslant k_i \leqslant N_\Delta - 1$ $(i = 1, 2, \dots, n)$, we set up the difference equation

$$a(x,\Delta)u = f, \tag{4}$$

which is obtained from (1) by replacing D^α by Δ^α. Equations (3) and (4) define an explicit difference scheme and make it possible to determine the function u successively on all layers $x_0 = k_0\Delta x_0$, $2m+2 \leqslant k_0 \leqslant M_\Delta + m$, if the values of u from Ω_Δ are periodically continued to the whole layer $x_0 = \text{const}$.

For the solution of (3), (4) the following holds:

Theorem 1. If $\varkappa \leqslant \varkappa_0$, the inequality

$$(\Delta x)^n\sum_{\Omega_\Delta}\sum_{|\gamma|\leqslant m+1,\ \gamma_0\leqslant m}|\Delta^\gamma u|^2\Big|_{x_0=t\Delta x_0} \leqslant \tag{5}$$

$$\leqslant C\left\{(\Delta x)^n\sum_{\Omega_\Delta}\sum_{i=0}^{m}\sum_{|\gamma|\leqslant m+1-i}|\Delta^\gamma\varphi_i|^2 + \Delta x_0\sum_{m+1}^{M_\Delta-1}(\Delta x)^n\sum_{\Omega_\Delta}\left(f^2+\sum_{i=1}^{n}|\Delta_i f|^2\right)\right\}$$

holds for $t = m+1, m+2, \dots, M_\Delta - 1$, where $\varkappa_0$ is a sufficiently small constant depending on the coefficients of the operator $a(x, D)$.

Here $\sum_{\Omega_\Delta}$ denotes summation over all points of Ω_Δ with fixed $x_0 = t\Delta x_0$; $\sum_{m+1}^{M_\Delta-1}\sum_{\Omega_\Delta}$ denotes summation over all points of Ω_Δ and all layers $x_0 = (m+1)\Delta x_0, \dots, (M_\Delta - 1)\Delta x_0$.

Scheme II. Define the function v as follows. On the layers $x_0 = k_0 \Delta x_0,\ 0 \le k_0 \le m$, prescribe v by formula (3). For the points $x = \{k_0 \Delta x_0, k_1 \Delta x, \dots, k_n \Delta x\}$ with $0 \le k_0 \le M_\Delta - 1,\ 0 \le k_i \le N_\Delta - 1\ (i = 1, 2, \dots, n)$, form the difference equation

$$a(x, \Delta) v = f. \tag{6}$$

Equations (6), together with (3) and the periodicity conditions, constitute an implicit difference scheme for the successive determination of v on all layers $x_0 = k_0 \Delta x_0,\ m + 1 \le k_0 \le M_\Delta + m$, if the obtained values of v are extended periodically from Ω_Δ to the whole layer $x_0 = \text{const}$. The solvability of these equations follows from inequality (7) of Theorem 2.

Theorem 2. For the solutions v of Scheme II the inequalities

$$(\Delta x)^n \sum_{\Omega_\Delta} v^2 \Big|_{x_0 = t\Delta x_0} \le C \left\{ (\Delta x)^n \sum_{\Omega_\Delta} \sum_{i=0}^{m} \sum_{|\gamma| \le m-i} |\Delta^\gamma \varphi_i|^2 + \Delta x_0 \sum_{0}^{M_\Delta - 1} (\Delta x)^n \sum_{\Omega_\Delta} f^2 \right\} \tag{7}$$

hold for $t = 0, 1, \dots, M_\Delta + m$;

$$(\Delta x)^n \sum_{\Omega_\Delta} \sum_{|\gamma| \le m+1,\ \gamma_0 \le m} |\bar{\Delta}^\gamma v|^2 \Big|_{x_0 = t\Delta x_0} \le C \Bigg\{ (\Delta x)^n \sum_{\Omega_\Delta} \sum_{i=0}^{m} \sum_{|\gamma| \le m+1-i} |\Delta^\gamma \varphi_i|^2 + \Delta x_0 \sum_{0}^{M_\Delta - 1} (\Delta x)^n \sum_{\Omega_\Delta} \Bigg(f^2 + \sum_{i=1}^{n} \tag{8}$$

for $t = 0, 1, \dots, M_\Delta - 1$ and $\chi \le R$, where R is an arbitrary positive constant.

Scheme III. The function w is defined by formula (3) on the layers $x_0 = k_0 \Delta x_0,\ 0 \le k_0 \le 2m + 1$. For the points $x = \{k_0 \Delta x_0, k_1 \Delta x, \dots, k_n \Delta x\}$ with $m + 1 \le k_0 \le M_\Delta - 1,\ 0 \le k_i \le N_\Delta - 1\ (i = 1, 2, \dots, n)$, form the equation

$$a(x, \widetilde{\Delta}) w = f. \tag{9}$$

Equations (3), (9), together with the periodicity conditions, constitute an implicit difference scheme for the successive determination of w on the layers $x_0 = k_0 \Delta x_0,\ 2m + 2 \le k_0 \le M_\Delta + m$, if w is extended from Ω_Δ periodically to the whole layer $x_0 = \text{const}$. The solvability of these equations follows from inequality (10) of Theorem 3.

Theorem 3. The solution w of Scheme III satisfies, for $\chi \le R$, the inequalities

$$(\Delta x)^n \sum_{\Omega_\Delta} w^2 \Big|_{x_0 = t\Delta x_0} \le C \left\{ (\Delta x)^n \sum_{\Omega_\Delta} \sum_{i=0}^{m} \sum_{|\gamma| \le m-i} |\Delta^\gamma \varphi_i|^2 + \Delta x_0 \sum_{m+1}^{M_\Delta - 1} (\Delta x)^n \sum_{\Omega_\Delta} f^2 \right\}$$

for $t = 0, 1, \dots, M_\Delta + m$;

$$(\Delta x)^n \sum_{\Omega_\Delta} \sum_{|\gamma| \le m+1,\ \gamma_0 \le m} |\widetilde{\Delta}^\gamma w|^2 \Big|_{x_0 = t\Delta x_0} \le C \Bigg\{ (\Delta x)^n \sum_{\Omega_\Delta} \sum_{i=0}^{m} \sum_{|\gamma| \le m+1-i} |\Delta^\gamma \varphi_i|^2 + \Delta x_0 \sum_{m+1}^{M_\Delta - 1} (\Delta x)^n \sum_{\Omega_\Delta} \Bigg(f^2 + \sum_{i=1}^{n} \tag{11}$$

for $t = m+1, \dots, M_\Delta - 1$ and any positive constant R.

The constants C appearing in Theorems 1-3 depend on the maxima of the moduli of the coefficients of the operator $a(x, D)$ and their Lipschitz constants, on R or χ_0, but do not depend on $f,\ \varphi_i,\ \Delta x_0,\ \Delta x$.

The main point in the proof of these theorems is the choice of a difference separating operator and the derivation, for difference quotients, of formulas analogous to the integration-by-parts formulas for derivatives. These formulas are based on the identities

$$\begin{aligned}
&[wv + (\overline{w}v)]_{x_0} = 2\,(\Delta_0 v J_2 w + \Delta_0 w J_2 v) && \text{for schemes I and III;}\\
&[wv]_{x_0} = v_{x_0} J_1 w + w_{x_0} J_1 v = \Delta_0 v J_1 w + \Delta_0 w J_1 v && \text{for scheme II;}\\
&\sum_{\Omega_\Delta} \Delta_i v \cdot w = -\sum_{\Omega_\Delta} \Delta_i w \cdot v \quad (i = 1, 2, \dots, n) && \text{for all schemes,}
\end{aligned}$$

where v, w are functions on the grid that are periodic in x'.

The separating operator is taken in the form $J_2 b(x, \Delta) u$, $J_1 b(x, \overline{\Delta}) v$, $J_2 b(x, \widetilde{\Delta}) w$, respectively, for schemes I, II, III. In all other respects the proof of Theorems 1-3 is a modification of Gårding' s arguments $(^2)$.

4. **Existence of a solution of problem (1), (2).** Let now $f(x) \in H^{0,1}(Q_T)$, $\varphi_i(x') \in H^{m+1-i}(\Omega)$ $(i = 0, 1, \dots, m)$, $f^\nu(x) \in C^\infty$, $\varphi_i^\nu(x') \in C^\infty$, and suppose that

   $$\|f - f^\nu\|_{H^{0,1}(Q_T)} \to 0, \qquad \|\varphi_i^\nu - \varphi_i\|_{H^{m+1-i}(\Omega)} \to 0$$

   as $\nu \to \infty$. Let $\Delta^\nu x_0, \Delta^\nu x$ be a sequence of steps tending to zero as $\nu \to \infty$, such that

   $$\|f^\nu_{\Delta^\nu} - f\| \to 0, \qquad \|(\Delta_j f^\nu)_{\Delta^\nu} - D_{jf}\| \to 0, \qquad \|(\Delta^\gamma \varphi_i^\nu)_{\Delta^\nu} - D^\gamma \varphi_i\| \to 0$$

   for $j = 1, 2, \dots, n;\ i = 0, 1, 2, \dots, m,\ |\gamma| \le m + 1 - i$; here $f^\nu_{\Delta^\nu}$ (analogously $(\varphi_i^\nu)_{\Delta^\nu}$) is the function constructed according to item 2 for f^ν, considered only on the grid with steps $\Delta^\nu x_0, \Delta^\nu x$, and $\|\ \|$ is the norm in $L_2(Q_T)$ or $L_2(\Omega)$.

Let now u_ν, v_ν, w_ν be functions on the grid with steps $\Delta^\nu x_0, \Delta^\nu x$, defined respectively by schemes I, II, III with initial functions equal to φ_i^ν $(i = 0, 1, \dots, m)$, and with right-hand side f^ν.

Theorem 4. *The functions* $(u_\nu)_{\Delta^\nu}$ *for* $x \leq x_0$ *and* $(v_\nu)_{\Delta^\nu}, (w_\nu)_{\Delta^\nu}$ *for* $x \leq R$ *converge weakly in* $L_2(Q_T)$*, as* $\nu \to \infty$*, to a function* $u \in H^{m+1}(Q_T)$ *satisfying equation (1) almost everywhere in* Q_T*, periodic in* x' *with period* 2π*, and assuming in the mean the values (2). For it the inequality*

$$\int_{Q_T} \sum_{|\alpha| \leq m+1} |D^\alpha u|^2 \, dx \leq C \left\{ \int_\Omega \sum_{i=0}^{m} \sum_{|\alpha| \leq m+1-i} |D^\alpha \varphi_i|^2 \, dx' + \int_{Q_T} \left(f^2 + \sum_{i=1}^{n} |D_i f|^2 \right) dx \right\}. \tag{12}$$

Moreover, the functions $(\Delta^\alpha u_\nu)_{\Delta^\nu}$*,* $(\Delta^\alpha v_\nu)_{\Delta^\nu}$*,* $(\widetilde{\Delta^\alpha} w_\nu)_{\Delta^\nu}$*,* $|\alpha| \leq m+1$*, converge weakly in* $L_2(Q_T)$ *to* $D^\alpha u$*. The constant* C *in (12) depends only on the coefficients of the operator* $a(x, D)$*.*

The proof of Theorem 4 is based on Theorems 1-3 and on S. L. Sobolev' s imbedding theorems.

Remark 1. The uniqueness of the solution constructed in Theorem 4 follows from Theorem 4.1 of (^{2}).

Remark 2. If the coefficients $a_\alpha(x)$, the right-hand sides f, and the initial functions φ_i are sufficiently smooth, one can show that the solution obtained in Theorem 4 belongs to H^{m+1+r} (r arbitrary natural) and, consequently, for sufficiently large r is a classical solution of problem (1), (2).

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Note: Figure translations are in progress. See original paper for figures.