MATHEMATICS

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1961

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Abstract

Full Text

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ON A MODIFICATION OF THE ITERATIVE METHOD WITH MINIMAL RESIDUALS FOR SOLVING NONLINEAR OPERATOR EQUATIONS

(Presented by Academician S. L. Sobolev on 15 VII 1960)

1. Let $P(x)$ be a twice differentiable (in the Fréchet sense) operator from the real Hilbert space H into the same space. For solving the equation

$$P(x) = 0 \tag{1}$$

we consider iterative methods of the type

$$x_{n+1} = x_n + \varepsilon_n y_n, \qquad n = 0, 1, \dots, \tag{2}$$

where $x_0 \in H$ is a known initial approximation to the solution of equation (1), ε_n are real numbers, and $\{y_n\}$ is some sequence of elements of the space H.

Since $P(x)$ is twice differentiable, we have

$$\|P(x_{n+1})\| \le \|P(x_n) + P'(x_n)(x_{n+1} - x_n)\| + \frac{1}{2}\|P''(\bar{x}_n)\|\,\|x_{n+1} - x_n\|^2, \tag{3}$$

where $\bar{x}_n = x_n + \tau_n(x_{n+1} - x_n)$, $0 < \tau_n < 1$.

Choose ε_n so that, for fixed y_n,

$$\|P(x_n) + P'(x_n)(x_{n+1} - x_n)\|^2 = \|P(x_n) + \varepsilon_n P'(x_n) y_n\|^2$$

is minimal. It is easy to verify that the minimum value is attained for

$$\varepsilon_n = -\frac{(P(x_n),\, P'(x_n) y_n)}{\|P'(x_n) y_n\|^2}. \tag{4}$$

In this case

$$\|P(x_n) + P'(x_n)(x_{n+1} - x_n)\|^2 = \|P(x_n)\|^2 - \frac{(P(x_n),\, P'(x_n)y_n)^2}{\|P'(x_n)y_n\|^2}. \tag{5}$$

2. Choose $y_n = P(x_n)$. Then, for solving equation (1), we obtain the method

$$x_{n+1} = x_n - \frac{(P(x_n),\, P'(x_n)P(x_n))}{\|P'(x_n)P(x_n)\|^2}\, P(x_n), \tag{6}$$

which, in the case of a linear operator equation, gives the method with minimal residuals considered by M. A. Krasnosel' skii and S. G. Krein (^{1}).

Concerning the convergence of method (6), the following theorem holds (cf. (^{2})):

Theorem 1. *Suppose the following conditions are satisfied:*

1°. $\|P(x_0)\| \le \delta_0$;

2°. For all $x \in S(x_0, r)^1$, where $r = \dfrac{M\delta_0}{1-q}$, the following estimates hold:

a) $\|P'(x)\| \le A$;

b) $\|P''(x)\| \le B$;

c) $(P'(x)h, h) \ge M^{-1}\|h^2\|$ for all $h \in H$ $(M > 0)$.

3°. $q = \sqrt{1 - b^{-1} + \frac{1}{2}a_0} < 1$, where $b = M^2A^2$, $a_0 = M^2B\delta_0$.

Then equation (1) has in the sphere $S(x_0, r)$ a unique solution x^*, to which the sequence $\{x_n\}$ obtained by method (6) converges, and the estimates

$$\|x^* - x_n\| \le M\|P(x_n)\| \le M\delta_0 q^n. \tag{7}$$

hold.

Proof. Since $\|P''(\bar{x}_0)\|\|x_1 - x_0\|^2 \le BM^2\delta_0^2 = a_0\delta_0$, taking into account (5) and the assumptions of the theorem, we obtain from (3)

$$\|P(x_1)\| \le \left(\sqrt{1 - b^{-1}} + \tfrac{1}{2}a_0\right)\|P(x_0)\| = q\|P(x_0)\|.$$

This means that there exists a constant δ_1 satisfying the inequalities $\|P(x_1)\| \le \delta_1 \le q\|P(x_0)\| \le q\delta_0 < \delta_0$.

It is easy to verify that $*$ $S(x_1, r_1) \subset S(x_0, r)$, where $r_1 = \dfrac{M\delta_1}{1-q_1}$, $q_1 = \sqrt{1 - b^{-1} + \frac{1}{2}a_1} < q_1$, $a_1 = M^2B\delta_1$. Thus all the assumptions are satisfied at x_1, and we may continue computing successive approximations. By mathematical induction we obtain, for all $n = 0, 1, \ldots$,

$$\|P(x_{n+1})\| \leq \delta_{n+1} \leq q\|P(x_n)\|, \qquad \|x_{n+1} - x_n\| \leq M\|P(x_n)\|.$$

Using these inequalities, we obtain for all n and p

$$\|x_{n+p} - x_n\| \leq M(\|P(x_{n+p-1})\| + \cdots + \|P(x_n)\|) \leq \frac{M\delta_0}{1-q} q^n.$$

This proves the existence of the limit $\lim_{n\to\infty} x_n = x^* \in S(x_0, r)$. Since the operator $P(x)$ is continuous, then $\|P(x^*)\| = \lim \|P(x_n)\| \leq \delta_0 \lim q^n = 0$, i.e. x^* is a solution of equation (1). By virtue of condition 2° c) this solution is unique in the sphere $S(x_0, r)$. By the same condition,

$$\|P(x_n)\|\,\|x_n - x^*\| \geq |(P(x_n) - P(x^*), x_n - x^*)| = |(P'(\bar{x}_n)(x_n - x^*), x_n - x^*)| \geq M^{-1}\|x_n - x^*\|^2$$

$$(\bar{x}_n = x^* + \tau_n(x_n - x^*), \quad 0 < \tau_n < 1),$$

whence (7) follows.

3. If condition 2° c) of Theorem 1 is replaced by a weaker condition (cf. (2)), then we have:

Theorem 2. Let the following conditions be satisfied:

1°. $\|P(x_0)\| = \delta_0 \leq \bar{\delta}_0$.

2°. $(P'(x_0)h, h) \geq M_0^{-1}\|h\|^2$ for all $h \in H$ $(M_0 > 0)$.

3°. For all $x \in S(x_0, r)$, where

$$r = \frac{1}{B}\left(\frac{1}{M_0} - \frac{1}{M^*}\right)\frac{\delta_0}{\bar{\delta}_0} \qquad (M^* = \lim M_n \leq +\infty),$$

the estimates

$$\|P'(x)\| \leq A, \qquad \|P''(x)\| \leq B$$

hold.

4°. The quantities $a_0 = M_0^2 B\bar{\delta}_0$ and $b_0 = M_0^2 A^2$ are such that the sequence $\{a_n\} = \{M_n^2 B\bar{\delta}_n\}$, computed by means of the recurrence relations

$$M_{k+1} = \frac{M_k}{1 - M_k^2 B\bar{\delta}_k}, \tag{8}$$

$$\bar{\delta}_{k+1} = \bar{\delta}_k\left(\sqrt{1 - (M_k^2 A^2)^{-1}} + \tfrac{1}{2} M_k^2 B\bar{\delta}_k\right)$$

converges (so that $a_n < 1$ for all n).

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* The symbol $S(x_0, r)$ denotes the sphere $\|x - x_0\| \leq r$.

Then equation (1) has in the sphere $S(x_0, r)$ a solution x^*, to which the sequence $\{x_n\}$ obtained by method (6) converges, and the estimates

$$\|x^* - x_n\| \leq \frac{2M_n\delta_n}{1 + \sqrt{1 - 2M_n^2 B\delta_n}} < 2M_n\delta_n, \tag{9}$$

hold, where $\delta_n = \|P(x_n)\|$, and M_n are defined recursively by formulas (8). If $M^* < \infty$ or $\bar{\delta}_0 > \delta_0$, then the solution is unique in the sphere $S(x_0, r)$.

Theorem 2 is proved essentially in the same way as Theorem 3 in paper (2), taking into account relations (3) and (5). To obtain estimates (9), we use Taylor' s formula

$$(P(x^*), h) = (P(x_n) + P'(x_n)(x^* - x_n) + {}^1\!/_2 P''(x_n + \tau_n(x^* - x_n))(x^* - x_n), h)$$

$$(0 < \tau_n < 1)$$

in the case

$$h = \overline{[P'(x_n)]^{-1}}(x^* - x_n).$$

Hence we obtain the inequality

$${}^1\!/_2 M_{nB}\|x^* - x_n\|^2 - \|x^* - x_n\| + M_n\delta_n \geq 0,$$

from which (9) follows.

Verification of the fulfillment of the conditions of Theorem 2 is facilitated by

Theorem 3. If $a_0 b_0 \leq {}^1\!/_9$, then condition 4° of Theorem 2 is fulfilled.

Proof. By virtue of the recurrence relations (8) and the condition of the present theorem,

$$a_{nb} n \leq \cdots \leq a_1 b_1 \leq a_0 b_0 \quad (b_k = M_k^2 A^2).$$

The assertion follows from this.

4. We choose

$$y_n = \overline{P'(x_n)} P(x_n),$$

where $\overline{P'(x)}$ is the operator adjoint to the linear operator $P'(x)$. Then we obtain the method

$$x_{n+1} = x_n - \frac{\|\overline{P'(x_n)} P(x_n)\|^2}{\|P'(x_n)\overline{P'(x_n)} P(x_n)\|^2} \overline{P'(x_n)} P(x_n). \tag{10}$$

On the convergence of method (10), the following theorems are valid:

Theorem 4. Suppose the following conditions are fulfilled:

1°. $\|P(x_0)\| \leq \delta_0$.

2°. For all $x \in S(x_0, r)$, where

$$r = \frac{M\delta_0}{1-q},$$

the estimates hold:

a) $\|P'(x)\| \leq A$;

b) $\|P''(x)\| \leq B$;

c)

$$\|P'(x)h\| \geq M^{-1}\|h\|$$

and

$$\|\overline{P'(x)}\,h\| \geq M^{-1}\|h\|$$

for all $h \in H$ $(M > 0)$.

3°.

$$q = \frac{b-1}{b+1} + \frac{1}{2}a_0 < 1,$$

where $b = M^2A^2$, $a_0 = M^2B\delta_0$.

Then equation (1) has in the sphere $S(x_0, r)$ a solution x^*, to which the sequence $\{x_n\}$ obtained from (10) converges, and the estimates

$$\|x^* - x_n\| \leq \frac{M}{1-q}\|P(x_n)\| \leq \frac{M\delta_0}{1-q}q^n$$

hold.

Theorem 5. Suppose the conditions of Theorem 2 are fulfilled, except for condition 2° and relations (8), which are replaced respectively by the conditions:

$$\|P'(x_0)h\| \geq M_0^{-1}\|h\| \quad \text{and} \quad \|\overline{P'(x_0)}h\| \geq M_0^{-1}\|h\| \quad \text{for all } h \in H \ (M_0 > 0)$$

and by the relations

$$M_{k+1} = \frac{M_k}{1 - M_k^2 B\bar{\delta}_k}, \qquad \bar{\delta}_{k+1} = \bar{\delta}_k \left(\frac{M_k^2A^2 - 1}{M_k^2A^2 + 1} + \frac{1}{2}M^2B\bar{\delta}_k \right). \tag{11}$$

Then equation (1) has in the sphere $S(x_0, r)$ a solution x^*, to which the sequence $\{x_n\}$ obtained from (10) converges, and the estimates (9) hold, where $\delta_n = \|P(x_n)\|$ and M_n are defined recursively by formulas (11).

The weakenings in the hypotheses of Theorems 4 and 5, as compared with the hypotheses of Theorems 1 and 2, are obtained by virtue of the self-adjointness of the operator $P'(x)P'(x)$, since now, in order to estimate the last term in (5), we may use Theorem 2 of (3).*

Verification of the fulfillment of the conditions of Theorem 5 is facilitated by

Theorem 6. If

$$(b_0+1)(9-12a_0+8a_0^2-2a_0^3)a_0 \leqslant 4 \quad \text{and} \quad a_0 \leqslant {}^4\!/_9,$$

then condition 4° of Theorem 5 is satisfied.

Finally, let us note that other choices of the elements y_n may also be of some interest. For example, with the choices $y_n = P'(x_n)\overline{P'(x_n)}P(x_n)$, $y_n = \overline{P'(x_n)}P'(x_n)\overline{P'(x_n)}P(x_n)$, etc., theorems analogous to those given above hold. In particular, if $y_n = [P'(x_n)]^{-1}P(x_n)$, then we obtain Newton' s method.

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Received
14 VI 1960

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* In the case of a finite-dimensional space, the same estimate already follows from (4.10) in (1).