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Abstract

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ON AN INFINITELY DIFFERENTIABLE FUNCTION WITH PRESCRIBED VALUES OF THE DERIVATIVES AT A POINT

(Presented by Academician A. N. Kolmogorov, 17 XII 1960)

1. E. Borel in 1895 showed that, for a given numerical sequence $a_0, a_1, \ldots$, one can construct* in a neighborhood of zero an infinitely differentiable function $\psi(t)$ such that $\psi^{(n)}(0) = a_n,\ n \geqslant 0$. The question arose whether such a function can be constructed in a prescribed class defined by inequalities on the derivatives. For the class of analytic functions, where $|\psi^{(n)}(t)| \leqslant C^n n^n$, this is obviously possible (by means of the Taylor series). For quasianalytic classes $C(m_n)$ there is the theorem of Carleman (^{3}), showing how, from the values of the derivatives $\psi^{(n)}(0) = \xi_n$, to reconstruct uniquely a function $\psi(t) \in C(m_n)$; however, not for every numerical sequence $\xi = (\xi_n, n \geqslant 0)$ such that $|\xi_n| \leqslant AB^n m_n$ does there exist in the class $C(m_n)$ a function $\psi(t)$ such that $|\psi^{(n)}(t)| \leqslant A_1 B_1^n m_n$. The question of the possibility of construction in non-quasianalytic classes remained open. This question was posed to the author by G. E. Shilov.

In the present note, for a given numerical sequence $a_n,\ n = 0, 1, \ldots$, admitting the estimate $|a_n| \leqslant CB^n n^{n\beta}$, $\beta > 1$, for any $\varepsilon > 0$ there is constructed on the interval $[-1, 1]$ an infinitely differentiable function $\psi(t)$ such that

$$\max_{|t| \leqslant 1} |\psi^{(n)}(t)| \leqslant C_1[(1+\varepsilon)\Gamma_\beta B]^n n^{n\beta}, \quad n \geqslant 0, \quad \Gamma_\beta =$$

$$= (\cos \pi/2\beta)^{-\beta}.$$

Various consequences of this fact are given, as well as its refinements and generalizations.

2. Consider (cf. (^{4}), Ch. 4) the Banach spaces $C^{\beta B}$ and $L_2^{\beta B}$ of all infinitely differentiable functions on the interval $[-1, 1]$ such that

$$C^{\beta B} = \{\varphi : \ |\varphi|_B = \sup_{n,t} |\varphi^{(n)}(t)|/B^n n^{n\beta} < \infty\}$$

and

$$L_2^{\beta B} = \left\{\varphi : \ |\varphi|_{B,2} = \left(\sum_0^\infty \int_{-1}^1 |\varphi^{(k)}(t)|^2 \, dt/B^{2k} k^{2k\beta}\right)^{1/2} < \infty\right\};$$

the space $L_2^{\beta B}$ is Hilbert. Put $C^\beta = \bigcup_\beta C^{\beta B}$.

A functional f on the space $C^{\beta B}$ (or $L_2^{\beta B}$) will be called **concentrated at the point** t_0, if it vanishes on all functions $\varphi \in C^{\beta B}$ that are identically equal to zero in some neighborhood of the point t_0. For simplicity, in what follows we shall speak of the point $t_0 = 0$. With each functional f concentrated at zero we associate** the function

$$\Phi(s) =$$

$$= f\left\{e^{its} h\left(\frac{t}{\varepsilon}\right)\right\},$$

where $h(t)$ is any function*** from C^γ, $1 < \gamma < \beta$, equal to zero for $|t| \geqslant 1$ and to one for $|t| \leqslant 1/2$. The values $\Phi(s)$ do not depend on ε, since the functional f is concentrated at zero. Note that to the functionals $\delta_n : \delta_n(\varphi) = \varphi^{(n)}(0)$ there correspond the functions $\Delta_n(s) = (is)^n$.

Lemma 1**.** *Let f be a functional concentrated at zero on the space $C^{\beta B}$, and let $\Phi(s)$ be the function corresponding to it. The function* $\Phi(s) =$

$$= \sum_0^\infty f_n s^n$$

is entire, and its Taylor coefficients satisfy the inequalities

$$|f_n| \leq \frac{C_\delta (1+\delta)^n \|f\|}{B^n (n \cos \pi/2\beta)^{\beta n}}$$

for all $\delta > 0$, where C_δ are constants independent of n and f.

Proof. The family $\varphi_s(t) = e^{its} h(t)$ of elements of the space $C^{\beta B}$ depends analytically on the parameter s for all its complex values; therefore the function

$$\Phi(s) = f(\varphi_s)$$

is entire.

By the definition of $\Phi(s)$, for all $\varepsilon > 0$ the estimate

$$|\Phi(s)| \leq \|f\| \, \|e^{its} h(t/\varepsilon)\|_B$$

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* Other solutions of this question, see (^{2}).

** This function $\Phi(s)$ is in fact the Fourier transform of the functional f, cf. (^{6}), p. 225.

*** On the construction of such functions $h(t)$, see, for example, (^{5}), p. 105.

**** An analogous fact was noted in (^{7}), p. 103, which was also the starting point of our work; our proof is technically simpler.

holds. By the choice of the function h we have

$$|h^{(n)}(t)| \leq CA^n n^{n\gamma}, \qquad |h^{(n)}(t/\varepsilon)| \leq C(A/\varepsilon)^n n^{n\gamma};$$

moreover,

$$\max_{|t|\leq\varepsilon} |(e^{its})^{(n)}| = e^{\varepsilon|\tau|} r^n,$$

where $s = \sigma + i\tau,\ r = |s|$. Therefore

$$\left|\left\{e^{its} h\left(\frac{t}{\varepsilon}\right)\right\}^{(n)}\right| \leq e^{\varepsilon|\tau|} C \sum_{k=0}^{n} \binom{k}{n} r^k \left(\frac{A}{\varepsilon}\right)^{n-k} (n-k)^{(n-k)\gamma} \leq Ce^{\varepsilon|\tau|}\left(r + \frac{A}{\varepsilon} n^\gamma\right)^n.$$

Choose λ so that $\gamma < \lambda < \beta$. Then for $r \leq n^\lambda$ we have

$$\sup_{r\leq n^\lambda} \frac{\left(r + \frac{A}{\varepsilon} n^\gamma\right)^n}{B^n n^{n\beta}} \leq \sup_n \frac{\left(1 + \frac{A}{\varepsilon}\right)^n n^{\lambda n}}{B^n n^{n\beta}} = D < \infty,$$

$$\sup_{r>n^\lambda} \frac{\left(r + \frac{A}{\varepsilon} n^\gamma\right)^n}{B^n n^{n\beta}} \leq \sup_n \frac{\left(r + \frac{A}{\varepsilon} r^{\gamma/\lambda}\right)^n}{B^n n^{n\beta}}.$$

For any $\delta > 0$, when

$$r \geq R(\varepsilon, \delta; \gamma/\lambda)$$

the inequality

$$\left(r + \frac{A}{\varepsilon} r^{\gamma/\lambda}\right) \leq (1+\delta) r$$

is satisfied; hence for such r we have

$$\sup_n \frac{\left(r + \frac{A}{\varepsilon} n^\gamma\right)^n}{B^n n^{n\beta}} \leq \sup_n \frac{(1+\delta)^n r^n}{B^n n^{n\beta}} \leq \exp\left\{\frac{\beta}{e}\left(\frac{1+\delta}{B}\right)^{1/\beta} r^{1/\beta}\right\}.$$

(For the last inequality see (^{4}), p. 204.)

The inequalities obtained above show that, if the constants $C_{\varepsilon\delta}$ are chosen appropriately, then for all $\delta > 0$ and $\varepsilon > 0$ the inequalities

$$\|e^{its} h(t/\varepsilon)\|_B \leq C_{\varepsilon\delta} \exp\{\varepsilon|\tau| + Kr^{1/\beta}\}$$

and

$$|\Phi(s)| \leq \|f\|\, C_{\varepsilon\delta} \exp\{\varepsilon|\tau| + Kr^{1/\beta}\}$$

hold, where

$$K = \frac{\beta}{e}\left(\frac{1+\delta}{B}\right)^{1/\beta}.$$

Thus the function $\Phi(s)$ is entire, of first order of minimal type, and on the real axis its order of growth is $1/\beta < 1$ and its type

$$K = \frac{\beta}{e}\left(\frac{1+\delta}{B}\right)^{1/\beta}$$

in the estimate $|\Phi(s)|$ does not depend on ε. Then, by the refined Phragmén-Lindelöf theorem,* in the whole complex plane the estimate

$$|\Phi(s)| \leq D_1\|f\| \exp\{\Gamma_\beta K r^{1/\beta}\}$$

is valid, where

$$\Gamma_\beta = (\cos\pi/2\beta)^{-\beta}.$$

The Cauchy inequalities for the Taylor coefficients and the relation (see (^{4}), p. 259, formula 5)

$$\inf_r\{r^{-n}\exp[Mr^{1/\beta}]\} \leq C\left(M\frac{e}{\beta}\right)^{\beta n}\left(\frac{1}{n}\right)^{\beta n}$$

give

$$|f_n| \leq D_\delta\left(\frac{1+2\delta}{B}\Gamma_\beta\right)^n n^{-n\beta}\|f\|, \qquad n \leq 0.$$

The lemma is thereby proved.

3. It is not difficult to verify that, for $B < B_1 < B_2$, the inclusions

$$C^{\beta B} \subset L_2^{\beta B_1} \subset C^{\beta B_2}$$

hold, and the corresponding operators are continuous. Hence it is clear that the assertion of Lemma 1 is also true for the spaces $L_2^{\beta B}$.

Denote by $K(m_n)$ the Banach space of sequences

$$\xi = (\xi_n,\ n \geq 0)$$

with norm

$$\|\xi\| = \sup_n \frac{|\xi_n|}{m_n}.$$

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* See (^{8}), p. 70, Theorem 22; compare with the corollary on p. 71.

Theorem 1. For every $\varepsilon > 0$ there exists a linear continuous operator $L : K(B^n n^\beta) \to C^{\beta,\Gamma_\beta(1+\varepsilon)}$ such that $(L\xi)^{(n)}(0) = \xi_n,\ n \geq 0$.

Proof. Consider the space $L_2^{\beta,\Gamma_\beta(B+\varepsilon/2)} = X$ and the closed subspace $E_0 \subset X'$ of functionals concentrated at zero. By Lemma 1, for $f \in E_0$ we have

$$|f_n| \leq D\|f\|\left(\frac{1}{B+\varepsilon/4}\right)^n n^{-n\beta}, \quad n \geq 0;$$

therefore, for every $a = (a_n) \in K(B^n n^\beta)$, the formula

$$A(f) = \sum_0^\infty (-i)^n f_n a_n$$

defines a linear continuous functional on E_0; indeed,

$$|A(f)| \leq \sum_0^\infty D\|f\|(B + \frac{\varepsilon}{4})^{-n} n^{-n\beta} B^n n^\beta \|a\| = C_\varepsilon \|f\| \cdot \|a\|.$$

The space X' is Hilbert; let P_0 be the operator of orthogonal projection in it onto E_0. Put, for $g \in X'$, $\hat{A}(g) = A(P_0 g)$, and define the operator $L : K \to X$ by the formula $\hat{A}(g) = g(La)$ for all $g \in X'$, which is possible since a Hilbert space is reflexive. The operator L, so defined, is linear and continuous (its norm does not exceed C_ε), and for all $n \geq 0$, since the functionals δ_n (derivatives of the delta-function) lie in E_0, for the function $\varphi(t) = (La)(t)$ we have

$$\varphi^{(n)}(0) = \delta_n(\varphi) = \hat{A}(\delta_n) = A(\delta_n) = (-i)^n i^n a_n = a_n.$$

Thus the constructed operator L satisfies all the conditions of the theorem.

The constant $\Gamma_\beta = (\cos \pi/2\beta)^{-\beta}$ occurring in the formulation of Theorem 1 is sharp in the sense that the following holds:

Theorem 1a. For any $\varepsilon > 0$ there exists a sequence $(\xi_n) \in K(B^n n^\beta)$ for which there is no function $\varphi(t)$ in the space $C^{\beta, \Gamma_\beta(B-\varepsilon)}$ such that $\varphi^{(n)}(0) = \xi_n, \ n \geq 0$.

Indeed, assuming the contrary, one can arrive at the assertion of Lemma 1, even with a more precise estimate of the coefficients of the function $\Phi(s)$:

$$|f_n| \leq \frac{A_\delta (1+\delta)^n \|f\|}{(B-\varepsilon)^n (n \cos \pi/2\beta)^{\beta n}}, \quad \delta > 0,$$

and from these estimates–to the conclusion that every entire function of order $F(s)$ with type $B_1 = B^{1/\beta}$ on the real axis has, in the whole plane, type not exceeding $(B-\varepsilon)^{1/\beta}(\cos \pi/2\beta)^{-1}$. Examples of concrete functions (cf. (^{8}), pp. 70-72) show that this assertion is false. Such is the scheme of the proof of Theorem 1a.

Lemma 1 and Theorem 1 are also valid in the multidimensional case. In what follows, for simplicity of formulation, we restrict ourselves to the two-dimensional case.

Let $K(m_{nl})$ be the space of doubly indexed sequences

$$\xi = (\xi_{nl}, n, l \geq 0)$$

with norm

$$\|\xi\| = \sup_{n,l} \frac{|\xi_{nl}|}{m_{nl}}.$$

Theorem 1b. For arbitrary B and $\varepsilon > 0$ there exists a linear continuous operator

$$L : K(B^{n+l} n^\beta l^\beta) \to C\{[(1+\varepsilon)\Gamma_\beta B]^{n+l} n^\beta l^\beta\} = C_2^{\beta,\Gamma_\beta B(1+\varepsilon)},$$

such that

$$D_t^n D_s^l (L\xi)\big|_{0,0} = \xi_{nl}.$$

4. Now, after a linear operator has been constructed for extending functions in the classes C^β from a point, one can extend functions from closed sets according to Whitney' s scheme (^{9}) (see also (^{10}), § 3), with some refinements.

Theorem 2. Let F be an arbitrary closed set lying inside the unit square. One can construct a linear continuous operator L from $C_2^{\beta B}$ to $C_2^{\beta,\Gamma_\beta(B+\varepsilon)}$ such that, for $\psi = L\varphi$,

$$D_t^n D_s^l \psi = D_t^n D_s^l \varphi$$

on F for all $n, l \geq 0$, and

$$\sup_{\substack{|t|,|s|\leq 1\\ n,l}} \frac{|D_t^n D_s^l \psi(t,s)|}{(\Gamma_\beta(B+\varepsilon))^{n+l} n^\beta l^\beta} \leq K_1 \sup_{\substack{(t,s)\in F\\ n,l}} \frac{|D_t^n D_s^l \varphi(t,s)|}{B^{n+l} n^\beta l^\beta}.$$

From Theorem 2 and the results of (^{11}) (Theorem 4) it follows*:

Theorem 3. *A linear functional g on C_2^β, if it is concentrated on a closed set F lying inside the unit square, admits the representation*

$$g(\varphi) = \sum_{n,l} \int_F D_t^n D_s^l \varphi(t,s)\, d\mu_{nl},$$

where μ_{nl} are measures on F, and for every B

$$\sum_{nl} \operatorname{Var} \mu_{nl} \cdot B^{n+l} n^{n\beta} l^{l\beta} < \infty.$$

Theorem 2 also makes it possible to extend to the case of spaces S_α^β (see (^{4}), pp. 210 and 282) Hörmander' s results ((^{10}), §4, 5) on division by a polynomial.

5. For sequences m_n, $n \geqslant 0$, of a more general nature than $n^{n\beta}$, $\beta > 1$, one can prove the following proposition:

Theorem 4. *Let a sequence of positive numbers m_n, $n \geqslant 0$, satisfy the following conditions: 1) the ratios $\mu_n = m_n/m_{n-1}$, $\mu_0 = 1$, tend monotonically to infinity; 2) the series $\sum \mu_n^{-1}$ converges, or, equivalently,*

$$\int_1^\infty \frac{\mu(t)}{t^2}\, dt < \infty,$$

where $\mu(t) = \sup\{n : \mu_n \leqslant t\}$. *Let the sequence* p_n, $n \geqslant 0$, *be defined by the relations*

$$p_n^{-1} = \inf\left\{r^{-n} \exp\left[r \int_1^{\infty} \frac{\mu(t)\,dt}{t(r+Dt)}\right]\right\},$$

where $D < 1$.

*Then there exists a linear continuous operator L from the Banach space of sequences $K(p_n)$ into the Banach space of infinitely differentiable functions $C(m_n)$** such that $(L\xi)^{(n)}(0) = \xi_n$ for all $n \geqslant 0$ and $\xi \in K$.*

The proof of this theorem is essentially the same as that of Theorem 1; in obtaining the analogue of Lemma 1, some estimates from (^{7}) are used.

In conclusion—a few words on the functional dimension df (for the definition see (^{12}), p. 22) of the spaces S_α^β.

Theorem 5. *The functional dimension*

$$df\{S^\beta[-1,1]\} = \beta.$$

In the case of several variables

$$df\{S^{\beta_1\beta_2\ldots\beta_n}[-1,1]^n\} = \sum_{i=1}^{n} \beta_i, \quad df\{S_{\alpha_1\alpha_2\ldots\alpha_n}^{\beta_1\beta_2\ldots\beta_n}\}$$

$$= \sum_{i=1}^{n} (\alpha_i + \beta_i).$$

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* In an imprecise form and without proof this fact was noted in $(^{11})$, Theorem 5.

** This space is defined analogously to the space $C^{\beta B}$: the sequence $B^n n^{n\beta}$ is replaced by m_n.

Note: Figure translations are in progress. See original paper for figures.

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