MATHEMATICS

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

M. I. Vishik

BOUNDARY-VALUE PROBLEMS FOR QUASILINEAR STRONGLY ELLIPTIC SYSTEMS OF EQUATIONS HAVING DIVERGENCE FORM

(Presented by Academician S. L. Sobolev on 7 I 1961)

Theorems are proved on the solvability and uniqueness of the solution of boundary-value problems for a certain class of quasilinear systems of differential equations. Roughly speaking, these are systems for which the corresponding equations in variations form a strongly elliptic operator (1). In order that the problems under consideration be solvable for arbitrary data and in an arbitrary domain, it is assumed, as in the linear case, that the quadratic form of the operator in variations is positive. In the first part of the note we give the results obtained in the most general form, and therefore the restrictions imposed have a functional form; in the second part sufficient algebraic conditions are given under which the solvability and uniqueness theorems hold (see (2)).

1. In a domain $D \subset R^n$ with boundary Γ there is given the system:

$$L(u) \equiv \sum_{|\alpha|,|\gamma| \leq m} (-1)^{|\alpha|} D_\alpha A_\alpha(x, D_\gamma u) = h(x), \tag{1}$$

where $x = (x_1, x_2, \dots, x_n)$, $D_\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $D_0 = E$ (the identity operator). The functions $u = (u^1, \dots, u^N)$, $h = (h^1, \dots, h^N)$, $A_\alpha(\ ) = (A_\alpha^1(\ ), \dots, A_\alpha^N(\ ))$ take vector values in R^N, and $A_\alpha(x, 0) = 0$. Thus, (1) is a system of N equations with N unknowns $u^i(x)$. One seeks a solution of (1) under the boundary conditions of the first boundary-value problem:

$$u|_\Gamma = \varphi_0(x), \dots, \partial^{m-1} u / \partial n^{m-1}|_\Gamma = \varphi_{m-1}(x), \qquad x \in \Gamma, \tag{2}$$

or under other boundary conditions (see Sec. 2).

For simplicity of notation we assume that the operator $L(u)$ (1) has the same order $2m$ with respect to all functions $u^1, \dots, u^N$. With obvious modifications everything carries over to the case when, with respect to u^k, it has order $2m_k$.

By a solution of the problem (1), (2) we understand a function u satisfying on Γ the conditions (2), for which

$$[A_\alpha(x, D_\gamma u), D_\alpha v] = [h, v] \tag{3}$$

for every function v satisfying the homogeneous conditions (2) on Γ; [,] is the usual scalar product of vector functions; repeated indices $\alpha, \beta, \ldots$ are everywhere to be summed as in (1). Obviously, the space V_0 of functions v and the space U, in which the solution u is sought, are chosen depending on the order of growth of $A_\alpha(\ ,\ )$ in the second arguments, in such a way that the integrand expressions in (3) are summable (see examples in Sec. 3). It is assumed that $V_0 \subset U$. For example, V_0 and U are spaces of type $W_p^{(l)}$ (see [3]), such that $A_\alpha(x, D_\gamma u) \in \mathcal{L}_p$, and $D_\alpha v \in \mathcal{L}_q$, $1/p + 1/q = 1$. We shall restrict ourselves only to the case of such spaces V_0 and U. We assume that for every function $w \in U$, or $w \in U' \subset U$, $\overline{U}' = U$, the quadratic form in $v \in V_0$, or $v \in V_0'$, $\overline{V}'_0 = V_0$,

$$A(w; v, v) = \sum_{|\alpha|, |\beta| \le m} [A_{\alpha\beta}(x, D_\gamma w) D_\beta v, D_\alpha v] \tag{4}$$

in a certain sense positive, for example:

$$A(w; v, v) \ge c^2 \left(\sum_{j,\ |\alpha|=m} \left[\left| D_\alpha w^j \right|^{\delta_j} D_\alpha v^j, D_\alpha v^j \right] + \cdots \right), \quad \delta_j \ge 0^*, \tag{I_1}$$

where the terms corresponding to the lower-order terms may be absent; $A_{\alpha\beta}(\ ) = \partial A_\alpha(\ )/\partial(D_\beta w)$, for fixed α and β, is a matrix of order N; or

$$A(w; v, v) \ge c^2 \sum_{|\beta|=m} \left[\left(1 + \sum |D_\alpha w|^2 \right)^{-\delta/2} D_\beta v, D_\beta v \right], \tag{I_2}$$

where $-\delta > -1$. More general conditions on $A(\ )$, under which our conclusions are valid, are given in Example 4 of §3. Obviously, $A(w; v, v)$ is the quadratic form of the variation of the operator L at the "point" w.

Putting in both sides of (I_1), respectively (I_2), $w = f + tv$ and integrating with respect to t from 0 to 1, we obtain:

$$I(v) \equiv [A_\alpha(x, D_\gamma(f + v)) - A_\alpha(x, D_\gamma f), D_\alpha v] \ge \tag{5}$$

$$\ge c^2 \sum \left[\left(|D_\alpha v^j|^{\delta_j} + |D_\alpha f^j|^{\delta_j} \right) D_\alpha v^j, D_\alpha v^j \right] + \cdots \quad (j = 1, \ldots, N;\ |\alpha| = m),$$

respectively

$$I(v) \geq c_1^2 \left[\left(1 + \sum |D_\alpha f|^2 + \sum |D_\alpha v|^2\right)^{-\delta/2} D_\beta v, D_\beta v \right]. \tag{6}$$

Consequently, in case (I_1) $I(v)$ has, with respect to $D_\alpha v^j$, order of growth not less than $2 + \delta_j$, and in case (I_2) not less than $2 - \delta > 1$.

Conditions (II) reduce to the fact that, under differentiation of $A_\alpha(\ )$ with respect to $D_\gamma v$, the order of growth decreases by one, and under partial differentiation with respect to x the order does not increase. For brevity we do not formulate these obvious restrictions here in general form, but give them only as applied to examples (§3). The principal one among the conditions (II) is the following:

$$\left|[\xi A_{\alpha\beta}(x, D_\gamma(f+z)) D_\beta v, D_\omega z]\right| + \left|[\xi A_{\alpha\beta}(\ ) D_\omega z, D_\alpha v]\right| \leq$$

$$\leq \varepsilon A(f+z; v, v) + MI(z) + C \quad (|\omega| \leq m,\ |\beta| \leq m), \tag{7}$$

where $\varepsilon > 0$ is arbitrarily small; $M = M(\varepsilon)$ and C are constants independent of v and z; f is fixed; ξ is a function vanishing on Γ together with several derivatives.

Our last condition: there is an exponent $p_\alpha > 1$ such that

$$\|A_\alpha(x, D_\gamma(f+v))\|_{p_\alpha} \leq CI(v) + M \tag{III}$$

for $v \in V_0$. Obviously, in verifying the fulfillment of this condition one should use embedding theorems.

We shall call system (1) **strongly elliptic and, moreover, definite** under the boundary conditions of the first boundary-value problem if conditions (I), (II), (III) are satisfied.

Theorem 1. *If the quasilinear system* (1) *is strongly elliptic and, moreover, definite, then for any right-hand side* $h \in H^{**}$ *problem* (1), (2) *is solvable, i.e., there exists a function* $u \in U$ *satisfying* (2) *and* (3).

Let $f(x)$ be some function from U satisfying conditions (2). Then $z = f - u \in V_0$, $u = f + z$. According to (3), the problem reduces to finding, for a given function $f(x)$ replacing conditions (2), a function $z \in V_0$ such that

$$[A_\alpha(x; D_\gamma(f+z)), D_\alpha v] = [h, v] \tag{8}$$

* Note that the summation on the right in (I_1) need not be over all α with $|\alpha| = m$.

** Here, as H, one may certainly take W_q^{-s} with the corresponding q, and in a number of cases one may take $H = \mathcal{L}_q$ or $H = \mathcal{L}_q^{(-s)}$ (see (2′)).

for $v \in V_0$. It is natural to call it the **projection** of $f(x)$ onto V_0. We prove the existence of this projection by means of the following Galerkin (moment) method. Let $\{v_i(x)\}$, $v_i \in V_0$, be such a system of smooth functions that the system $(M - \psi\Delta)v_i$ is complete in V_0 (see Lemma 2). We seek an approximate value of z in the form $z_n = \sum C_{in} v_i$, where the C_{in} are determined from the system of equations

$$[(M - \Delta\psi)L(f + z_n), v_j] = [h, (M - \psi\Delta)v_j], \qquad j = 1, \dots, n, \tag{9}$$

where $\psi(x) > 0$, $x \in D$, is a smooth function vanishing, together with several derivatives, on the boundary Γ; $M > 0$; Δ is the Laplace operator.

Lemma 1 (basic). *For any smooth $v \in V_0$ the estimate holds:*

$$[P(L(f + v) - L(f)), v] \geq c^2 I(v) + [\psi A_{\alpha\beta}(f + v) D_\beta D_i v, D_\alpha D_i v] + C, \tag{10}$$

where M is a sufficiently large constant, $P \equiv M - \Delta\psi$.

Hence, and from the lemma of the note ([2]), we derive the solvability of system (9). Further, using condition (I_1) or (I_2) and (10), we prove, analogously to ([2]), the convergence of a subsequence z_n to z; moreover, in D almost everywhere $A_\alpha(x, D_\gamma z_n) \to A_\alpha(x, D_\gamma z)$, and in the metric $\mathcal{L}_{p_\alpha}$ they have uniformly bounded norms. Consequently, according to (9), z satisfies (8) with v replaced by $(M - \psi\Delta)v_j$.

It follows from this that the function z found satisfies (8) for any $v \in V_0$. The existence of the "projection" z, and hence the solvability of problems (1), (2), is proved.

Lemma 2. *For sufficiently large M and any right-hand side $g \in \mathcal{L}_2$, the equation (without boundary conditions)*

$$P_\ell^* v \equiv (M - \psi\Delta)v = g \tag{11}$$

is solvable, and in a unique way, in the class of functions with finite norm

$$\|v\| + \|\psi^{1/2} D_i v\| < +\infty, \qquad D_i = \partial/\partial x_i.$$

Theorem 2 (uniqueness). *If condition* (I_2) *is fulfilled, or condition* (I_1)*, in the right-hand side of which there is the term $[v, v]$, then the constructed solution of* *the strongly elliptic system (1) is unique in the space U. It is assumed that, for $u \in U$,*

$$A_\alpha(x, D_\gamma u) \in \mathcal{L}_{p_\alpha}, \quad D_\alpha u \in \mathcal{L}_{q_\alpha}, \quad 1/p_\alpha + 1/q_\alpha = 1$$

(see (2)*).*

2. Other boundary value problems

For simplicity we restrict ourselves to the second boundary value problem for systems of second order, i.e. in the domain D equation (1) is given with $m = 1$, and on the boundary Γ the condition

$$\sum_{|\alpha|=1}^{n} A_\alpha(x, D_\gamma u) \cos(n, x_\alpha)\big|_\Gamma = \varphi = (\varphi^1, \dots, \varphi^N). \tag{12}$$

The solution of problem (1), (12) is defined by the relation

$$[A_\alpha(x, D_\gamma u) - A_\alpha(x, D_\gamma f), D_\alpha v] = [h, v] - [A_\alpha(x, D_\gamma f), D_\alpha v], \tag{13}$$

where the function $f(x)$, $x \in D+\Gamma$, satisfies condition (12) on Γ; v is any element of the space V of type $W_p^{(1)}$, not subject to any conditions on Γ. The condition of strong ellipticity of problem (1), (12) is analogous to conditions (I_1), (I_2), (II), (III) with V_0 replaced everywhere by V. We also find the solution of problem (1), (12) by means of an analogue of the projection method: $u = f + z$, $z \in V$. There hold existence and uniqueness theorems for the solution of problem (1), (12), analogous to Theorems 1 and 2 and proved analogously.

3. Sufficient algebraic conditions. Examples

The algebraic conditions given below are sufficient for the solvability and uniqueness of the solution of both the first and the second boundary value problems in any domain D.

1) We begin with the simplest polynomial case:

$$A_\alpha(x, D_\gamma u) = \sum a_{\alpha\gamma\delta}(x)(D_{\gamma_1} u^1)^{\delta_1} \cdots (D_{\gamma_N} u^N)^{\delta_N}, \qquad \gamma_i = (\gamma_{i1}, \dots, \gamma_{in}), \quad |\gamma_i| = m, \quad \delta_1 + \cdots$$

$\dots + \delta_N = 2l+1$. If the coefficients $a_{\alpha\gamma\delta}(x)$ are bounded in D and have bounded derivatives, and moreover

$$C^2(\ )(\ ) \ge (A_{\alpha\beta}(x, \eta_\gamma)\xi_\beta, \xi_\alpha) \ge c^2 \left(\sum |\eta_\gamma|^{2l}\right)\left(\sum |\xi_\alpha|^2\right), \tag{14}$$

then the system (1) under the conditions (2) is strongly elliptic, and Theorem 1 is valid. As U it suffices to take $W_p^{(m)}$, where $p = 2l+2$, $V_0 \subset U$. In the case of the second boundary-value problem the same is true if in equation (1) there is also, for example, a term $\omega^2(x)u$, $\omega^2(x) \geq \omega^2 > 0$. In this case there also holds the uniqueness theorem for the solution of both boundary-value problems.

2) Suppose that in equation (1) there are only the terms $A_\alpha(x, D_\gamma u)$ with $|\alpha| = m$, $|\gamma| = m$. Further, for arbitrary η_γ, ξ_α,

$$C^2(\ ) \geq (A_{\alpha\beta}(x, \eta_\gamma)\xi_\beta, \xi_\alpha) \geq c^2((1 + \sum |\eta_\gamma|^2)^{-\delta/2}\xi_\alpha, \xi_\alpha), \tag{15}$$

where $-\delta > -1$, and $A_\alpha(\ )$ and $\partial A_\alpha/\partial x$ have, with respect to η, order of growth not exceeding $1-\delta$, while $A_{\alpha\beta}(\ )$ and $\partial A_{\alpha\beta}/\partial x$ have order not greater than $-\delta$. Then problem (1), (2) is strongly elliptic, and Theorems 1 and 2 hold. Here $U = W_p^{(m)}$, where $p = 2 - \delta$. If in (1) there is also a term $\omega^2 u$, then the second boundary-value problem also has these properties.

Remark. The solutions u constructed in examples 1) and 2) have in D derivatives up to order $m+1$ and, in the case $m = 1$, satisfy (1) almost everywhere in D (see [2]). We note that a single elliptic equation of second order of the form (1) has been studied by other methods under conditions of type (14) in a number of works (see, for example, [4–7] and the literature cited there).

3) Analogous assertions are also valid when equation (1) contains terms of lower order. In this case it is sufficient that only the right-hand inequality (15) (or the inequality analogous to (I_1)) be satisfied, with summation in the middle term carried out over all $|\alpha| \leq m$, $|\beta| \leq m$, while on the right it is sufficient to sum over $|\alpha| = m$. Instead of the left-hand inequality, the subordination conditions (II) and (III) are imposed.

4) The conditions (I_1), (I_2), (14), (15) can be considerably weakened. For constructing a solution, i.e. for the validity of Theorem 1, it is sufficient that there merely exist functions $F_\alpha = F_\alpha(\eta_\gamma)$, $|\alpha| = |\gamma| = m$, which map, topologically, the Euclidean space of all $\{\eta_\gamma\}$, $|\gamma| = m$, onto the space of all $\{F_\alpha\}$, $|\alpha| = m$, and that

$$(A_{\alpha\beta}(x, \eta_\gamma)\xi_\beta, \xi_\alpha) \geq c^2 \sum_{\alpha,\beta} |F_{\alpha\beta}(\eta_\gamma)\xi_\beta|^2, \qquad F_{\alpha\beta} = \frac{\partial F_\alpha}{\partial \eta_\beta},$$

and that the conditions mentioned at the end of example 3) be fulfilled.

5) If the operator $L(u) = L^0(u) + L^1(u)$, where L^0 is a linear strongly elliptic operator with constant coefficients $A^0_{\alpha\beta}$, and

$$\sum (A^0_{\alpha\beta}\xi^p_\beta\xi^q_\alpha\bar{\xi}, \bar{\xi}) > 0 \quad \text{for } \xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n} \neq 0,\ |\bar{\xi}| = 1,$$

while $L^1(u)$ is a quasilinear strongly elliptic operator, and moreover definite, then there holds the theorem of existence and uniqueness of the solution of problem (1), (2). This example shows that, in the case of the first boundary-value problem, the functional conditions of item 1 are less restrictive than the conditions of item 3.

Received
4 I 1961

REFERENCES CITED

[1] M. I. Vishik, *Matem. sborn.*, **29** (71), 615 (1951).
[2] M. I. Vishik, *DAN*, **137**, No. 3 (1961).
[3] S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.
[4] S. N. Bernstein, *UMN*, **8**, 32 (1941).
[5] O. A. Ladyzhenskaya, *DAN*, **120**, No. 5 (1958).
[6] T. B. Solomyak, *DAN*, **127**, No. 2 (1959).
[7] N. N. Ural' tseva, *DAN*, **130**, No. 6 (1960).

Note: Figure translations are in progress. See original paper for figures.