ON SUPERCONVERGENCE AND NONCONTINUABILITY OF FUNCTIONAL SERIES

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Abstract

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S. V. VARGANOVA and Yu. F. KOROBEINIK

ON SUPERCONVERGENCE AND NONCONTINUABILITY OF FUNCTIONAL SERIES

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Studying lacunary power series, Ostrovskii showed that these series superconverge in a neighborhood of every regular point on the boundary of the circle of convergence ([1]). He also proved the converse theorem, according to which every power series possessing a partial sequence uniformly convergent in a neighborhood of some regular point on the boundary of the circle of convergence is a series of lacunary structure. Thus, the existence of a superconvergent partial sequence completely determines the lacunary structure of the series. It should be noted that Ostrovskii' s converse theorem is proved much more difficultly than the direct one. Ostrovskii' s works were developed in Burion' s papers ([2]), which obtained important results on superconvergent power series. Recently there have appeared papers ([3–6]) in which the superconvergence and noncontinuability of functional series were studied (mainly, series in polynomials).

In the present note Ostrovskii' s direct theorem is carried over to the case of functional series of a rather general nature; the result obtained includes, as special cases, some results from ([3,5,6]). In addition, with the aid of certain characteristics introduced by M. A. Evgrafov ([7]), an assertion is proved which generalizes Ostrovskii' s converse theorem.

For the formulation of the results obtained we shall need certain concepts and definitions.

Let $u(z) \geq 0$ be a continuous function of the variable z such that the equations $u(z) = \rho$ represent rectifiable curves C_ρ, surrounding the origin, which, as ρ decreases, contract to the point $z = 0$; moreover the curve C_{ρ_1} lies inside the curve C_{ρ_2} if $\rho_1 < \rho_2$. As in ([7]), by D_ρ we denote the simply connected domain lying inside C_ρ, and by the domain $R_1 < u(z) < R_2$ we shall mean the ring-shaped domain between C_{R_1} and C_{R_2}.

We shall say that a system of functions $\{\varphi_n(z)\}$ possesses property (S) in the domain $R_1 < u(z) < R_2$, if for every $\varepsilon > 0$ and any r_1 and r_2, $R_1 < r_1 < r_2 < R_2$, one can indicate a constant A and a number N such that for all $n > N$ and all z in the domain $r_1 \leq u(z) \leq r_2$

$$|\varphi_n(z)| < A\,(u(z) + \varepsilon)^n. \tag{1}$$

It is obvious that if the system of functions $\{\varphi_n(z)\}$ satisfies condition (S), then for every ρ from (R_1, R_2)

$$\overline{\lim}_{n\to\infty} \sqrt[n]{\max_{z\in C_\rho} |\varphi_n(z)|} \leq \rho. \tag{2}$$

It is easy to show that condition (S) is satisfied by the regular systems of M. A. Evgrafov ([7]), almost regular systems ([6]), and also by the quasi-power-like internally continuable bases of M. G. Khaplanov ([8]).

Functional series

$$f(z) = \sum_{n=0}^{\infty} a_n \varphi_n(z) \tag{3}$$

with coefficients satisfying the condition

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \frac{1}{R} \qquad (R_1 < R < R_2), \tag{4}$$

converges uniformly in every closed domain lying inside the domain $R_1 < u(z) < R$.

The series (3) is called a series of lacunary structure if it can be split into two series in such a way that one series has gaps of relative length bounded below, while the other has a larger radius of convergence.

It follows from the definition that $a_\nu = a'_\nu + a''_\nu$, where

$$\overline{\lim_{\nu\to\infty}} \sqrt[\nu]{|a'_\nu|} = \frac{1}{R},$$

$$\overline{\lim_{\nu\to\infty}} \sqrt[\nu]{|a''_\nu|} = \frac{1}{r} \qquad (r > R) \tag{5}$$

and the coefficients $a'_\nu = 0$ for $m_k < \nu < n_k$ $(k = 1, 2, ...)$, $n_k > \lambda m_k$, $\lambda > 1$.

In the paper [7], M. A. Evgrafov introduced the concept of a first reduced system. Namely, he calls a system of functions $\{\varphi_n(z)\}$, analytic in some neighborhood of the origin, a first reduced system if $\varphi_n(z) = \sum_{k=n}^{\infty} a_{n,k} z^k$ $(a_{n,n} \neq 0,\ n = 0, 1, 2, ...)$. If the first reduced system is regular and forms a basis in D_ρ for all ρ from (R_1, R_2), then we shall call such a system a regular first reduced basis.

Theorem 1. *Let the functions $\{\varphi_n(z)\}$ be analytic in the domain $R_1 < u(z) < R_2$ and satisfy condition (S) there. In order that the series (3), under condition (4), be a series of lacunary structure, it is necessary, and in the case where $\ln u(z)$ is a superharmonic function in $R_1 < u(z) < R_2$ and $\{\varphi_n(z)\}$ is a regular* *first reduced basis, also sufficient, that there exist a subsequence $S_{n_k}(z)$ of the series (3) possessing the following property: in some bounded domain G, exterior to C_R, the inequalities*

$$\frac{1}{n_k}\ln|S_{n_k}(z)| < \psi(z) + \varepsilon_{n_k}(z), \qquad k = 1, 2, \dots,$$

hold, where $\psi(z)$ is a continuous function in G satisfying there the condition

$$\psi(z) < \ln\frac{u(z)}{R},$$

and $\varepsilon_{n_k}(z) \to 0$ uniformly in this domain.

Putting, in particular, $\varphi_n(z) = z^n$, and $u(z) = |z|$, we obtain Bourion' s theorem [2].

Theorem 2 *(an analogue of Ostrowskĭ s direct theorem). If the series (3), under condition (4), is a series of lacunary structure, i.e. the relations (5) hold, the system of functions $\{\varphi_n(z)\}$ satisfies condition (S), and $\ln u(z)$ is a superharmonic function in the domain $R_1 < u(z) < R_2$, then the subsequence of partial sums*

$$S_{m_k}(z) = \sum_{n=0}^{m_k} a_n\varphi_n(z)$$

converges uniformly in a neighborhood of each point of regularity of $f(z)$ lying on C_R.

Theorem 2 contains the result of paper ([3]), as well as Theorem 2 from ([6]). From Theorem 2 there follows an assertion generalizing the well-known theorem of Alamar ([1]):

Theorem 3. Let the functions $\{\varphi_n(z)\}$ form a regular system in D_{R_2}, or an almost regular system in the domain $R_1 < u(z) < R_2$, and let $\ln u(z)$ be a superharmonic function in this domain. Then, if the indices of the nonzero coefficients of the series (3), under condition (4), form a sequence $\{n_m\}$ in which $n_{k+1} > (1+\theta)n_k,\ \theta > 0$, then the curve C_R is a cut for the function $f(z)$.

Theorem 4. Suppose that the functions $\varphi_n(z)$ are analytic in D_{R_2} and constitute a regular first reduced basis. Suppose further that $\ln u(z)$ is a superharmonic function in $R_1 < u(z) < R_2$. Then every series (3), under condition (4), having a subsequence $S_{n_k}(z)$ of partial sums that is overconvergent in a neighborhood of some point of C_R, is a series of lacunary structure.

Remark. Theorem 4 will remain true if (with the other indicated conditions satisfied) the series (3) has not an overconvergent partial subsequence $S_{n_k}(z)$, but a partial subsequence $\{S_{n_k}(z)\}$ bounded in some sufficiently small neighborhood of a point $z_0 \in C_R$.

We shall call a point z_0 on the contour C_R a generalized singular point for the function $f(z)$, defined by equality (3) under condition (4), if no subsequence of the series

$$\sum_{n=0}^{\infty} a_n \varphi_n(z)$$

converges uniformly in a neighborhood of the point z_0. Then the following result, which follows from Theorem 4, is valid:

Theorem 5. If the series (3), under condition (4), in functions of a regular first reduced basis is not a series of lacunary structure, and $\ln u(z)$ is a superharmonic function in $R_1 < u(z) < R_2$, then all points on C_R are generalized singular points of the function $f(z)$.

Rostov-on-Don
State University

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Note: Figure translations are in progress. See original paper for figures.