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Abstract

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ON THE THEORY OF THE DISTRIBUTION OF COLLECTIONS OF PRIME NUMBERS WITH GIVEN DIFFERENCES BETWEEN THEM

(Presented by Academician I. M. Vinogradov on 16 II 1961)

§ 1. For integers $m \geqslant 1$; $u_1, \dots, u_m$, let $\pi(x; u_1, \dots, u_m)$ denote the number of primes p in the interval $(0, x)$ such that $p + u_1, \dots, p + u_m$ are also primes. The function introduced is of the greatest importance in the theory of prime numbers; in particular, it expresses Goldbach' s binary problem, the problem of twin primes, and many other unsolved problems. In the general case, as regards the quantity $\pi(x; u_1, \dots, u_m)$, only results having the character of upper estimates are so far known.

In the present paper, on the basis of I. M. Vinogradov's method of trigonometric sums (1), asymptotic laws are established for $\pi(x; u_1, \dots, u_m)$, with extraction also of remainder terms of growth, valid asymptotically for all nontrivial systems of numbers $u_1, \dots, u_m$.

Theorem 1. Let $m \geqslant 1$ be an integer; $\gamma_m(p)$ be the number of residue classes to which $0, u_1, \dots, u_m$ belong modulo a prime p; $0 \leqslant \beta \leqslant 2$. Then

$$\pi(x; u_1, \dots, u_m) = \prod_p \frac{p - \gamma_m(p)}{p\left(1 - \dfrac{1}{p}\right)^{m+1}} \left(\int_2^x \frac{dz}{\ln^{m+1} z} + O\left(\frac{x}{\ln^{m+c} x} \right) \right) + \beta$$

for every system of distinct numbers $u_1, \dots, u_m$ of the interval $[1, X]$, $X = x/(\ln x)^{m+c}$, with the exception of no more than $\ll X^m/(\ln X)^M$ of them, where $M > 0$, $c > 1$ are arbitrary constants and the $\ll$, O quantities do not depend on x.

Remark 1. Since the number of systems of distinct numbers $u_1, \dots, u_m$ in the interval $[1, X]$ is a quantity of order X^m, this thereby establishes the validity of the Hardy-Littlewood hypothesis (2) concerning $\pi(x; u_1, \dots, u_m)$ for almost all systems $u_1, \dots, u_m$.

I. M. Vinogradov' s method also makes it possible to obtain a considerably more general result, characterizing the distribution of collections of prime numbers $p, p+u_1, \dots, p+u_m$ in arithmetic progressions. Here the difference of the progression may grow in a known manner together with the number of its terms. Namely:

Theorem 2. Let $m \geqslant 1$; $D \geqslant 1$ be integers; $\pi(x; D, u_1, \dots, u_m)$ be the number of collections of prime numbers $p, p+u_1, \dots, p+u_m$ in the interval $(0, x)$, belonging respectively to the progressions $Dn+l, Dn_1+l_1, \dots, Dn_m+l_m$, with $D \leqslant (\ln x)^A$, A an arbitrary positive constant; $1 \leqslant l, l_1, \dots, l_m \leqslant D$ and coprime to D.

Then for every system of numbers satisfying the natural conditions:

1) $1 \leqslant u_1, \dots, u_m \leqslant X; \quad X = x/(\ln x)^{m+c}$;

2) $u_\nu \equiv l_\nu - l \pmod{D} \quad (\nu = 1, \dots, m)$;

3) $u_s \neq u_t$, if $s \neq t$, excluding no more than $\ll X^m/D^m(\ln X)^M$ of them, where $M > 0$, $c > 1$ are arbitrary constants, the equality holds

$$\pi(x; D, u_1, \dots, u_m) = \frac{1}{D} \prod_p \frac{p - \gamma_m(p, D)}{p\left(1 - \frac{1}{p}\right)^{m+1}} \left(\int_2^x \frac{dz}{\ln^{m+1} z} + O\left(\frac{x}{\ln^{m+c} x}\right)\right) + \beta,$$

where $0 \leq \beta \leq 2$; p runs through the sequence of prime numbers; $\gamma_m(p, D)$ is the number of solutions of the congruence

$$(D\xi + l)(D\xi + l + u_1) \dots (D\xi + l + u_m) \equiv 0 \pmod{p}$$

and $\ll, O$ are quantities depending only on m, M, C, and A.

§ 2. Let us outline the main points of the proof of Theorem 2. Introduce additional notation: $p, p_1, \dots, p_m$ are prime numbers; χ is a character mod$D$; $\nu = 1, \dots, m$;

$$S_\nu(\alpha_\nu) = \sum_\chi \overline{\chi}(l_\nu) \sum_{3 \leq p_\nu \leq x} \chi(p) \exp(2\pi i \alpha_\nu p_\nu);$$

$$S(\alpha_1, \dots, \alpha_m) = \sum_\chi \overline{\chi}(l) \sum_{3 \leq p \leq x} \chi(p) \exp\left(-2\pi i p \sum_{\nu=1}^m \alpha_\nu\right),$$

where $\overline{\chi}$ is the character conjugate to χ;

$$H(\alpha_1, \dots, \alpha_m) = S(\alpha_1, \dots, \alpha_m) \prod_{\nu=1}^m S_\nu(\alpha_\nu).$$

Next let a_ν, q_ν be integers; $a_\nu \le q_\nu$ and $(a_\nu, q_\nu) = 1$; $d_\nu = (q_\nu, D)$; $\delta_\nu = q_\nu/d_\nu$; $E(q_\nu) = 1$, if $(\delta_\nu, D) = 1$, otherwise $E(q_\nu) = 0$; g_ν is an integer such that $\delta_\nu g_\nu \equiv 1 \pmod{d_\nu}$; $V_\nu = \delta_\nu g_\nu$; μ is the Möbius function;

$$F_{a_\nu q_\nu}(\alpha_\nu) = E(q_\nu)\frac{\mu(\delta_\nu)}{\varphi(\delta_\nu)} \exp\left(2\pi i \frac{a_\nu V_\nu l_\nu}{q_\nu}\right) \sum_{n_\nu=3}^{\le x} \frac{1}{\ln n_\nu} \exp\left(2\pi i n_\nu \left(\alpha_\nu - \frac{a_\nu}{q_\nu}\right)\right),$$

$$F_{a_1 q_1,\dots,a_m q^m}(\alpha_1,\dots,\alpha_m) =$$

$$= E(q)\frac{\mu(\delta)}{\varphi(\delta)} \exp\left(-2\pi i \frac{aVl}{q}\right) \sum_{n=3}^{\le x} \frac{1}{\ln n} \exp\left(-2\pi i n \sum_{\nu=1}^{m} \left(\alpha_\nu - \frac{a_\nu}{q_\nu}\right)\right),$$

where a and q are determined by the relations

$$\frac{a_1}{q_1} + \dots + \frac{a_m}{q_m} = \frac{a}{q}, \qquad (a, q) = 1;$$

$\delta = q/d$; $d = (q, D)$; $E(q) = 1$ for $(\delta, D) = 1$ and zero otherwise. And, finally, put

$$N(\alpha_1,\dots,\alpha_m) = \sum_{q_1,\dots,q_m}^{[1,\Delta]} \sum_{a_1,\dots,a_m}^{q_1,\dots,q_m} F_{a_1 q_1,\dots,a_m q^m}(\alpha_1,\dots,\alpha_m) \prod_{\nu=1}^{m} F_{a_\nu q_\nu}(\alpha_\nu),$$

where $q_1,\dots,q_m$ independently run through the values of the integers in the segment $[1,\Delta]$, $\Delta = (\ln x)^\lambda$; λ is any constant > 0, and the numbers $a_1,\dots,a_m$ run through reduced residue systems, respectively, modulo $q_1,\dots,\mathrm{mod} q_m$.

In the indicated notation we obtain the first main lemma

$$\int_0^1 \cdots \int_0^1 |H(\alpha_1,\dots,\alpha_m) - N(\alpha_1,\dots,\alpha_m)|^2 \, d\alpha_1 \dots d\alpha_m \ll \frac{x^{m+2}}{(\ln x)^{M+(m+2)(m+c)}}, \tag{1}$$

the fundamental basis for whose derivation is formed by the results of I. M. Vinogradov on estimates of trigonometric sums with prime numbers and the law of prime numbers in progressions. Namely:

Lemma of I. M. Vinogradov. Let a, q be integers; $(a, q) = 1$;

$$D \ll (\ln x)^A; \qquad (\ln x)^\gamma < q \ll x(\ln x)^{-\gamma},$$

$$T = \sum_{\substack{3 \le p \le x \\ p \equiv l \pmod D}} \exp\left(2\pi i \frac{ap}{q}\right).$$

Then for any prescribed λ', under the condition that $\gamma \ge 64(\lambda' + A + 1)$,

$$|T| \ll x\left(D(\ln x)^{\lambda'}\right)^{-1}.$$

We shall use the law of distribution of prime numbers in progressions in the following form:

Lemma A. If, for $\nu = 1, \dots, m$, $q_\nu \ll (\ln x)^\gamma$; χ' is a character $\bmod Q \ll (\ln x)^{\gamma \cdot m}$,

$$S'(a_1, \dots, a_m) = \sum_{3 \le p \le x} \chi'(p) \exp\left(-2\pi i p \sum_{\nu=1}^{m} \left(\alpha_\nu - \frac{a_\nu}{q_\nu}\right)\right),$$

then, uniformly in Q,

$$S'(a_1, \dots, a_m) = \xi(\chi') \sum_{n=3}^{x} \frac{1}{\ln n} \exp\left(-2\pi i n \sum_{\nu=1}^{m} \left(\alpha_\nu - \frac{a_\nu}{q_\nu}\right)\right) + R;$$

$$|R| \ll x \exp\left(-\gamma_1 (\ln x)^{1/2}\right); \qquad \xi(\chi') = \begin{cases} 1, & \text{if } \chi' = \chi'_0, \\ 0, & \text{if } \chi' \ne \chi'_0. \end{cases}$$

Further, the left-hand side in (1) does not exceed the expression:

$$\sum_{\mathfrak{M}} \left|\varphi^{m+1}(D)\, \pi(x; D, u_1, \dots, u_m) - \Psi(x; D, u_1, \dots, u_m)\right|^2,$$

where the summation extends over all possible systems of numbers $u_1, \dots, u_m$ satisfying conditions 1)-3) of Theorem 2;

$$\Psi(x; D, u_1, \dots, u_m) = \sigma_\Delta(u_1, \dots, u_m; D)\, \Gamma(x; u_1, \dots, u_m);$$

where

$$\sigma_\Delta(u_1, \dots, u_m; D) = \sum_{\substack{q_1, \dots, q_m \\ [1, \Delta]}} G_{u_1, \dots, u_m}(q_1, \dots, q_m; D);$$

$$G_{u_1, \dots, u_m}(q_1, \dots, q_m; D) =$$

$$= \prod_{\nu=1}^{m} E(q_\nu)\frac{\mu(\delta_\nu)}{\varphi(\delta_\nu)} \sum_{\substack{a_1,\dots,a_m\\ q_1,\dots,q_m}} E(q)\frac{\mu(\delta)}{\varphi(\delta)} \prod_{t=1}^{m} \exp\left(2\pi i\,\frac{a_t}{q_t}(V_t l_t - Vl - u_t)\right),$$

$$\Gamma(x;u_1,\dots,u_m) = \sum_{\substack{n,n_1,\dots,n_m\\ [3,x]}} \sum_{u_1=n_1-n} \cdots \sum_{u_m=n_m-n} \frac{1}{\ln n\,\ln n_1 \cdots \ln n_m}.$$

The investigation of $\sigma_\Delta(u_1,\dots,u_m;D)$—a segment of the so-called singular series of the problem—requires considerable computations. Under the conditions of Theorem 2, one can obtain only that

$$\sigma_\Delta(u_1,\dots,u_m;D) = D^m \prod_{p\nmid D} \frac{p-\gamma_m(p,D)}{p\left(1-\frac{1}{p}\right)^{m+1}} + O\left(\frac{D^{2m}\tau(u)}{\Delta^{(1-\varepsilon)m}}\right), \qquad (2)$$

where u is the product of the distinct differences formed from the numbers $0,u_1,\dots,u_m$; $\tau(u)$ is the number of divisors of u; $\varepsilon > 0$ is an arbitrarily small constant.

The requirement that relation (2) be nontrivial imposes an additional condition on the systems of numbers $u_1,\dots,u_m$, namely that $\tau(u)$ not exceed some power of $\ln x$. Thus, under the conditions of the theorem, a number of systems of order $X^m/D^m(\ln X)^M$ is excluded.

For each system of numbers not excluded in this way from (2) and from the estimate of the function $\Gamma(x;u_1,\dots,u_m)$, we obtain the second main lemma:

$$\Psi(x;D,u_1,\dots,u_m) = \frac{\varphi^{m+1}(D)}{D} \prod_{p} \frac{p-\gamma_m(p,D)}{p\left(1-\frac{1}{p}\right)^{m+1}} \left(\int_2^x \frac{dz}{\ln^{m+1} z} + O\left(\frac{x}{\ln^{m+C} x}\right)\right).$$

Combining the first and second main lemmas (see (^{3})) gives Theorems 1 and 2.

In conclusion, we note that Theorem 2, the special cases and applications of which were indicated in ($^{3-5}$), can be generalized to progressions with unequal differences. The proof remains essentially the same.

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