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Abstract

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ON THE STRUCTURE OF SLOW MAGNETOHYDRODYNAMIC SHOCK WAVES IN THE PRESENCE OF BAROTROPY

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The question of the structure of oblique magnetohydrodynamic shock waves was considered earlier in papers ([1−3]). The problem of the structure of magnetohydrodynamic shock waves reduces to clarifying the question of which singular points of the system of ordinary differential equations describing one-dimensional stationary motions of a conducting gas can be connected by integral curves, whether such a connection is unique, and whether it depends on the magnitude of the dissipative coefficients. In paper ([2]) the problem of the structure of shock waves was considered in the presence of all dissipative coefficients: viscosity, thermal conductivity, and magnetic viscosity. It was shown that a system of 4 ordinary equations describing one-dimensional stationary motions of a gas has no more than 4 singular points S_1, S_2, S_3, S_4 (which are numbered in the order of increasing density at these points), and the character of the behavior of the integral curves in the neighborhood of these singular points was determined. It was shown that there always exists a unique integral curve effecting the transition $S_1 \to S_2$, which corresponds to a fast shock wave in the sense of ([4]). The pairs of points $S_1 \to S_3$, $S_1 \to S_4$, and $S_2 \to S_3$ are connected by integral curves not for all relations between the dissipative coefficients. These transitions correspond to shock waves unstable in the sense of ([4]).

The question of the structure of slow shock waves ($S_3 \to S_4$) in the presence of all dissipative coefficients remained open in ([2]). This is apparently connected with the mathematical difficulties of investigating integral curves in four-dimensional space, and also with the fact that in ([2]) an example was constructed in which the points S_3 and S_4 are not connected by an integral curve. However, as was found in ([3]), an error was made in constructing this example.

In papers ([1,3]) the structure of magnetohydrodynamic shock waves was investigated in the case when, of the dissipative coefficients, only the magnetic viscosity and the second kinematic viscosity are different from zero. In paper ([3]) it was shown that, within the framework of the indicated formulation, there always exists a unique integral curve connecting the points S_3 and S_4 and representing the structure of a slow shock wave, and the character of the connection of the other pairs of singular points was also clarified.

Below, under the assumption of barotropy, it will be proved that slow shock waves always possess a structure, i.e. that the points S_3 and S_4 are connected by a unique integral curve for arbitrary dissipative coefficients. The assumption of barotropy is convenient in that it reduces the problem to the study of integral curves in three-dimensional space; at the same time, however, some features of the four-dimensional nonbarotropic case are preserved (cf. the character of the singular points in ([2]) and in the present paper).

If the pressure is a function only of the specific volume,

$$p = p(V), \tag{1}$$

then the equation expressing the constancy of the energy flux serves to determine the internal energy of the medium, which does not influence the motion of the medium, since the pressure depends only on the specific volume.

Equality (1) can be ensured either by the properties of the medium, the internal energy of which must then have the form:

$$\mathcal{E}(V,s) = \mathcal{E}_1(V) + \mathcal{E}_2(s), \qquad p = -\partial\mathcal{E}/\partial V, \qquad T = \frac{\partial\mathcal{E}}{\partial s}, \tag{2}$$

or by virtue of properties of the process that ensure equalization of the temperature (infinite thermal conductivity, caused, for example, by radiative transfer of energy).

In what follows we shall assume that the function $p(V)$ satisfies the inequalities

$$dp/dV < 0, \qquad d^2p/dV^2 > 0. \tag{3}$$

The equations describing one-dimensional (along the x-axis) steady motions of a barotropic medium have the form

$$\frac{\nu_m}{4\pi}\frac{dH}{dx} = M\left(\frac{HV}{4\pi} - H_0^* v + E^*\right) \equiv F_H,$$

$$m_2\frac{dv}{dx} = M\left(v - H_0^* H\right) \equiv F_v, \tag{4}$$

$$m_1 M_2 \frac{dV}{dx} = M\left[p(V) + M^2 V + \frac{H^2}{8\pi} - P\right] \equiv F_V,$$

where M and P are the fluxes of mass and of the x-component of momentum; H and v are the components of the magnetic-field strength and of the velocity along the y-axis; $H_0^* = H_x/4\pi M = \text{const}$; $E^* = cE_y/4\pi M = \text{const}$; ν_m is the magnetic viscosity; m_1 and m_2 are the viscosity coefficients of the medium, with

$m_1 \geq 0$ and $m_2 \geq 0$. The constants H_0^* and E^* may, without loss of generality, be regarded as positive. The first equation expresses Ohm' s law, the second and third express the constancy of the fluxes of the x- and y-components of momentum.

Let us first of all study the surfaces obtained by equating to zero the right-hand sides of equations (4). The surface $F_v = 0$ is the plane $v = H_0^* H$, passing through the V-axis. The surface $F_V = 0$ is a cylindrical surface parallel to the v-axis and generated by the curve

$$p(V) + M^2 V + H^2/8\pi = P, \tag{5}$$

lying in the H, V plane. Under the assumptions (3) this curve is an oval curve symmetric with respect to the V-axis. The section of the surface $F_H = 0$ by the plane $v = \text{const}$ is the hyperbola

$$HV/4\pi = H_0^* v - E^*. \tag{6}$$

The surface $F_H = 0$ may be regarded as being composed of such hyperbolas. For $v > E^*/H_0^*$ it lies in the region $H > 0$, $V > 0$. The intersection of the surfaces $F_v = 0$ and $F_H = 0$ is the hyperbola

$$H\left(V - 4\pi H_0^{*2}\right) = -4\pi E^*, \tag{7}$$

lying in the plane $v = H_0^* H$, with horizontal asymptote $V = 4\pi H_0^*$, or $u \equiv MV = \sqrt{H_x^2 V/4\pi}$, where $\sqrt{H_x^2 V/4\pi}$ is the Alfvén velocity. The singular points of the system of equations (4) evidently lie at the intersection of the hyperbola (7) with the line described by equation (5) in the plane $v = H_0^* H$. In this case the points S_1 and S_2 lie above the asymptote, and the points S_3 and S_4 below it. The slow shock wave corresponds to the transition $S_3 \to S_4$.

The surfaces $F_H = 0$, $F_v = 0$, and $F_V = 0$ divide the space $H > 0$, $v > 0$, $V > 0$ into several regions. Let us denote by I–VIII the sets,

in which the following inequalities are satisfied:

I. $F_H > 0,\ F_v > 0,\ F_V > 0.$	V. $F_H < 0,\ F_v > 0,\ F_V > 0.$
II. $F_H > 0,\ F_v > 0,\ F_V < 0.$	VI. $F_H < 0,\ F_v > 0,\ F_V < 0.$
III. $F_H > 0,\ F_v < 0,\ F_V > 0.$	VII. $F_H < 0,\ F_v < 0,\ F_V > 0.$
IV. $F_H > 0,\ F_v < 0,\ F_V < 0.$	VIII. $F_H < 0,\ F_v < 0,\ F_V < 0.$

Here the sets II—VIII are regions; set I consists of two regions I_1 and I_2, the first of which adjoins the point S_3, and the second S_4 (see Fig. 1, where the sections $v = \text{const}$ are shown respectively for $v < v(S_4)$ (a), for $v(S_4) < v < v(S_3)$ (b), and for $v > v(S_3)$ (c)).

Fig. 1

Figure 1: Fig. 1

Let us now consider how the boundaries of regions I—VIII are intersected by integral curves. First of all, it is not difficult to see that if an integral curve leaves the region $H > 0,\ v > 0,\ V > 0$, then it cannot return to it. Indeed, through the plane $V = 0$ integral curves only enter. If an integral curve leaves through the plane $v = 0$, then, as x increases, it will always remain in the region where $dH/dx > 0$ and $dv/dx < 0$, and therefore cannot return. If an integral curve leaves through the plane $H = 0$ (which is possible for $v > E^*/H_0^*$), then subsequently it will be in the region where $dv/dx > 0,\ dH/dx < 0$, and cannot return to the region $H > 0,\ v > 0,\ V > 0$. Such integral curves cannot connect the singular points S_3 and S_4; therefore in what follows one may restrict oneself to studying the behavior of the integral curves only in the region $H > 0,\ v > 0,\ V > 0$.

Fig. 1

The direction of intersection of the surfaces $F_H = 0,\ F_v = 0$, and $F_V = 0$ by integral curves can be determined in a manner analogous to that used in ([2]), by considering the signs of the derivatives $dF_H/dx|_{F_H=0}$, $dF_v/dx|_{F_v=0}$, and $dF_V/dx|_{F_V=0}$ (see Fig. 1b, where arrows indicate the direction in which the integral curves pass through the surfaces $F_H = 0,\ F_v = 0$, and $F_V = 0$). It follows that, as x increases, integral curves can only leave regions II and VII; integral curves enter the region IV + VIII only from regions II and VII; and from the region I + V + VI integral curves can pass directly only into region III.

Let us consider the behavior of the integral curves in a neighborhood of the singular points of system (4). Applying the method of paper ([2]), one may conclude that at the point S_3 the eigenvalues satisfy the inequalities

$$\lambda_1 \leq 0 \leq \lambda_2 \leq \lambda_3,$$

whereas at the point S_4 the inequalities are

$$\lambda_1 \leq \lambda_2 \leq 0 \leq \lambda_3.$$

Thus, the integral curves leaving the point S_3 form a surface Σ_1, which in a neighborhood of S_3 is tangent to the plane passing through the eigenvectors corresponding to λ_2 and λ_3, while the integral curves entering S_4 form a surface Σ_2, tangent at S_4 to the plane passing through the eigenvectors corresponding to λ_1 and λ_2.

Studying the behavior of these surfaces in the neighborhood of S_3 and S_4, one can draw the following conclusions:

1. The surface Σ_1 in the neighborhood of S_3 passes through regions I, III, IV, VIII, VI and has no points inside regions II and VII.
2. The surface Σ_1 ($V = V_1(H, v)$) at the point S_3 satisfies the condition $\partial V_1/\partial H > 0$.
3. The surface Σ_2 in the neighborhood of S_4 passes through regions VII, IV, VIII and has no points inside regions III and V + VI.
4. The surface Σ_2 ($V = V_2(H, v)$) at the point S_4 satisfies the condition $\partial V_2/\partial H < 0$.

Since the integral curves everywhere leave regions II and VII, the surface Σ_1 cannot intersect the boundaries of these regions and, going from the point S_3 between these regions, must arrive at the point S_4, where II and VII approach S_4. The surface Σ_1 cannot terminate or split into parts, since there are no other singular points between S_3 and S_4. Thus, there exists at least one integral curve connecting the points S_3 and S_4.

We shall now prove that there exists only one such curve. First of all, let us show that only those integral curves can arrive at the point S_4 which, on leaving S_3, remain all the time in the region IV + VIII. Indeed, in II and VII there are no integral curves leaving S_3. Integral curves from region III cannot arrive at S_4, since they cannot pass from this region into other regions and cannot approach S_4 directly from it. If an integral curve for $v > v(S_4)$ lies in the region I + V + VI, then it cannot enter S_4, since in this region $dv/dx > 0$, and according to the preceding an integral curve can leave this region only into region III, whence it cannot get to S_4. An integral curve that has left S_3 cannot, for $v < v(S_4)$, find itself in the region I + V + VI, since for $v < v(S_4)$ this region borders only on regions II, VII, III; moreover, in regions II and VII there are no integral curves leaving S_3, while in III the integral curves enter.

We shall now show that from the region IV + VIII only one integral curve enters S_4. By virtue of equations (4), in the region IV + VIII the inequalities hold

$$\frac{\partial}{\partial V}\left(\frac{dH}{dv}\right) = \frac{\partial}{\partial V}\left(\frac{HV'/4\pi - H_0^* v + E^*}{v - H_0^* H}\right) < 0,$$

$$\frac{\partial}{\partial H}\left(\frac{dV}{dv}\right) = \frac{\partial}{\partial H}\left(\frac{P(V) + M^2 V + H^2/8\pi - P}{v - H_0^* H}\right) < 0.$$

From the first inequality it follows that on the surface Σ_1, which at the point S_3 satisfies the condition $\partial V_1/\partial H > 0$, there can be no point, as v decreases in the region IV + VIII, where $\partial V_1/\partial H = \infty$; and from the second inequality it follows that in the region IV + VIII) there can be no point where $\partial V_1/\partial H = 0$. Therefore everywhere in this region the surface Σ_1 satisfies the condition $\partial V_1/\partial H > 0$. Since at the point S_4 the surface Σ_2 satisfies the condition

$\partial V_2/\partial H < 0$, it is clear that the surfaces Σ_1 and Σ_2 have a unique line of intersection, which is the unique integral curve connecting S_3 and S_4.

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Note: Figure translations are in progress. See original paper for figures.