PARTIALLY ORDERED TOPOLOGICAL GROUPS

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

S. N. SLUGIN

PARTIALLY ORDERED TOPOLOGICAL GROUPS

(Presented by Academician S. L. Sobolev, 15 VII 1960)

1. Usually the topology in groups with a partial ordering is generated by it ([1]). Here a topology introduced "from outside" is considered. In the groups constructed, certain methods of two-sided estimation of the solution of a functional equation are studied.

2. In what follows X is an additive commutative group with zero θ. We first introduce an object in which the connection between convergence and inequality is expressed weakly, but is sufficient for the study of certain processes in which countable monotone sequences participate.

Definition 1. Let in the topological group ([2]) X, for some of its elements, the inequality relation $x > \theta$ be defined, with the following properties:
1) if $x > \theta$, then $x \neq \theta$;
2) if $x > \theta$, $y > \theta$, then $x + y > \theta$;
3) if $x_n > x_{n+1} \to \theta$, then $x_n > \theta$.
Then X will be called an **almost ordered group** (a Q-group). As usual, $x > y$, $y < x$ means $x - y > \theta$. Unlike a K-group ([1]), here the existence of the least upper bound of a bounded set is not required. An example of a Q-group in which not every set has a least upper bound is the space C, if the relation $x > \theta$ is defined as "inequality on the average"

$$\int_a^b x(t)\,dt > 0$$

or as $x(t_i) \geqq\neq 0$ (t_i fixed).

Schemes for constructing certain methods of two-sided approximations in an almost ordered Banach space are indicated in ([3]).

If X is a Q-group and Y is a topological subgroup (closed) ([2]) of the topological group X, then Y itself is a Q-group.

3. A closer connection between inequality and convergence can be obtained by generalizing the notion of a KB-lineal ([1]). A group differing from a K-group only in that the existence of a least upper bound is postulated only for finite sets will be called a K_0-group. We shall call a neighborhood

G of zero monotone if from the relations $|x| \leq |y|$, $y \in G$, it follows that $x \in G$. If all neighborhoods constituting a basis are monotone, then we shall call it monotone. For the monotonicity of G it is sufficient that it be symmetric ($-G = G$) and that the following conditions hold:

1) if $-y < x < y$, $y \in G$, then $x \in G$;

2) if $x, y \in G$, then $x \vee y \in G$.
   Here $x \vee y = \sup\{x, y\}$ ([1]).

Definition 2. A K_0-group that is a topological group with a monotone basis will be called a **semi-ordered topological group** (a KT-group). A KT-group is a Q-group, but in a KT-group other usual connections between convergence and inequality are also preserved, except for convergence of a monotone bounded sequence.

Definition 3. If a KT-group X is a K-group, and if from (o)-convergence ([1]) of a countable sequence there follows topological convergence (in the sense of the KT-group), then X will be called an (o)-complete KT-group.

Topological convergence follows from (o)-convergence if, from the relations $x_n > x_{n+1}$, $\inf\{x_n\} = \theta$, there follows the existence, for any G, of some $x_n \in G$. In an (o)-complete KT-group every monotonically increasing countable sequence bounded above has a limit.

Definition 4. A topological subgroup Y of a topological group X will be called a KT-subgroup $Y \subset X$ if, for $x, y \in Y$, the bound $(x \vee y)_X$ is the bound $(x \vee y)_Y$. A KT-subgroup is itself a KT-group.

Definition 5. Q- (or KT-) groups X and Z will be called Q- (or KT-) equivalent if a one-to-one correspondence $x = \varphi z$ is established between their elements, with φ and φ^{-1} additive, continuous, and positive ([1]).

4. In the Q-group X one can carry out a combination of Chaplygin-type linearization methods $((^4), theorem 4)$ and replacement of a linear equation $((^5), Ch. 14)$.

The set of all elements $x \in X$ for which $y \leqslant x \leqslant z$ will be called the interval $[y, z]$. Let the following conditions be satisfied:

a) the operator P maps $[x_0, \bar{x}_0] \subset X$ into the Q-group Y; $[\theta, \bar{x}_0 - x_0]$ is included in the domain of definition of an additive operator Γ, having a positive inverse on Y;

b) $P(\underline{x}_0) < \theta < P(\bar{x}_0)$, $\Delta P \equiv P(x + \Delta x) - P(x) \leqslant \Gamma \Delta x$ for $\Delta x > \theta$;

c) the processes

$$u_{n+1} = u_n - \Gamma^{-1} P(u_n), \tag{1}$$

where $u_0 = \underline{x}_0$ (or $u_0 = \bar{x}_0$), converge to a solution x^* of the equation $P(x) = \theta$;

d) there are topological subgroups $\widetilde{X} \subset X$, $\widetilde{Y} \subset Y$ and certain Q-groups $\bar{X}$, $\bar{Y}$, for which a Q-equivalence is established: $\tilde{\varphi}\widetilde{X} = \bar{X}$, $\psi\widetilde{Y} = \bar{Y}$;

e) the operator L maps Y into X, $\widetilde{Y}$ into $\widetilde{X}$; $\Gamma L \leqslant I$ on Y $(Iy \equiv y)$.

Define $\bar{L} = \varphi L \psi^{-1}$. The symbol $\bigvee$ will denote one of the relations $\leqslant, \geqslant$.

Theorem 1. *If conditions a)–e) are fulfilled, the corrections $\Delta x_n = x_n - x_{n+1}$ can be computed by the rule*

$$\Delta x_n = \varphi^{-1} \Delta \bar{x}_n, \qquad \Delta \bar{x}_n = \bar{L} \bar{z}_n \tag{2}$$

where $\bar{z}_n = \psi z_n$, $z_n \in \widetilde{Y}$, $\theta \bigvee z_n \bigvee P(x_n)$. If $x_0 = \underline{x}_0$ (or $= \bar{x}_0$), then $x_n \leqslant x_{n+1} \leqslant x^*$ (or $x_n \geqslant x_{n+1} \geqslant x^*$).

The process of the theorem can be accelerated if one uses the operators $\Gamma_n, \bar{\Gamma}_n$ of theorem 4 ([4]) instead of Γ.

5. A similar combination is possible for the method of alternating approximations (([6]), §2).

Let:

f) $\Delta P \geqslant \Gamma \Delta x$ for $\Delta x > \theta$; $\Gamma(\bar{x}_0 - \underline{x}_0) \geqslant P(\bar{x}_0)$, $\Gamma(\bar{x}_0 - \underline{x}_0) \geqslant -P(\underline{x}_0)$;

g) Γ maps X onto Y, $\widetilde{X}$ onto $\widetilde{Y}$.

Take $L = \Gamma^{-1}$.

Theorem 2. *If conditions a), c), d), f), g) are fulfilled, as the elements z_n one may take elements of $\widetilde{Y}$ satisfying, in Y, the inequalities*

$$\Gamma(x_n - x_{n-1}) \bigvee z_n \bigvee P(x_n) \quad (n = 0, \dots, \infty),$$

where $x_{-1} = \bar{x}_0$ (or x_0), if $x_0 = \underline{x}_0$ (or $\bar{x}_0$). Then the x_n, determined by the corrections (2), satisfy the inequalities

$$x_{n-2} \bigvee x_n \bigvee x^* \bigvee x_{n+1} \bigvee x_{n-1}.$$

6. In some cases the uniqueness of the solution and the convergence of the iterative processes (1) are established by topological methods. We give one sufficient criterion for convergence of these processes in an almost ordered group X.

Theorem 3. *For convergence to the solution x^* of the processes (1) under conditions a), b) or a), f), it is sufficient that X be a topological subgroup of an (o)-complete*

of a KT-group, and that there exist an additive operator Λ such that on $[x_0, \bar{x}_0] \subset X$ the conditions are satisfied: $\Lambda \Delta x \leq \Delta P$ (if b) is satisfied) or $\Lambda \Delta x \geq \Delta P$ (if f) is satisfied) for $\Delta x > \theta$, the iteration $(I - \Gamma^{-1}\Lambda)^n \to 0$ as $n \to \infty$, and the operator $\Gamma^{-1}\Lambda$ is continuous.

7. In conclusion we consider one method of constructing a two-sided estimate of a solution that does not require the use of topology. Let relations $>$ be defined in the (algebraic) groups X, Y. The operators Γ and P take X into Y. Let M be a subgroup of the group X; let K be the class of elements equivalent modulo M, i.e., if $x, x' \in K$, then $x - x' \in M$. Let Λ be an additive operator taking M into Y.

Theorem 4. *If it is known that the solution x^* of the equation $P(x) = \theta$ in the class K is positive, if $\Gamma > P$ (or $\Gamma < P$) in K, $\Gamma(x) - \Gamma(x') \equiv \Lambda(x - x')$, Λ^{-1} is positive on the image $\Lambda(M)$, and z is a solution of the equation $\Gamma(x) = \theta$ in K, then $x^* > z$ (or $x^* < z$).*

Here $\Gamma > P$ means, as usual, $\Gamma(x) > P(x)$ for $x > \theta$. As Γ and P one may take, for example, the operations $Ax - y$, $Bx - y$, where A, B are linear differential expressions, and A^{-1} is known. In this case M and K are sets of functions satisfying the given linear (homogeneous for M and, in general, nonhomogeneous for K) initial or boundary conditions.

Kazan State University
named after V. I. Ulyanov-Lenin

Received
22 XII 1959

CITED LITERATURE

1. L. V. Kantorovich, B. Z. Vulikh, A. G. Pinsker, *UMN*, **6**, no. 3 (43), 31 (1951).

2. L. S. Pontryagin, *Continuous Groups*, 1954.

3. S. N. Slugin, *Bul. Inst. Politehn. din Iași (R. P. Romînă)*, Ser. nouă, **4** (8), Fasc. 3-4, 81 (1958).

4. S. N. Slugin, *DAN*, **120**, no. 3, 472 (1958).

5. L. V. Kantorovich, G. P. Akilov, *Functional Analysis in Normed Spaces*, 1959.

6. S. N. Slugin, *Izv. Vyssh. uchebn. zaved., Matem.*, no. 6 (7), 244 (1958).

Note: Figure translations are in progress. See original paper for figures.