MATHEMATICS

V. F. DYACHENKO

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Abstract

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V. F. DYACHENKO

ON THE CAUCHY PROBLEM FOR QUASILINEAR SYSTEMS

(Presented by Academician M. V. Keldysh on 11 VII 1960)

Consider the hyperbolic system of quasilinear equations

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{1}$$

where $u = \{u_1, u_2, \dots, u_n\}$, $f = \{f_1, f_2, \dots, f_n\}$. Usually a generalized solution of system (1) is understood to be a piecewise-continuous function $u(x, t)$ satisfying the condition

$$\oint_\Gamma u\, dx - f(u)\, dt = 0 \tag{2}$$

for any contour Γ. As is known, with this definition the Cauchy problem for system (1) has, generally speaking, a nonunique solution. In the work of I. M. Gel' fand (^{1}) another definition of a generalized solution is proposed. Namely, instead of system (1) one considers the system with viscosity which approximates it,

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \varepsilon \frac{\partial}{\partial x} B(u) \frac{\partial u}{\partial x}, \tag{3}$$

where ε is a small parameter and $B(u)$ is an arbitrary positive-definite matrix. The function $u(x, t)$ is called a generalized solution of system (1) if it is the limit (as $\varepsilon \to 0$) of the solutions $u_\varepsilon(x, t)$ of system (3). We shall show that this definition also does not select a unique solution.

Consider the system of equations

$$\begin{gathered}
\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\cos\frac{3\pi}{2}\, u\right) = \varepsilon \frac{\partial}{\partial x}\left(b(u)\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\right), \\
\frac{\partial v}{\partial t} - \frac{\partial v^2}{\partial x} = \varepsilon \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right),
\end{gathered} \tag{4}$$

where

$$b(u) = 2 - \frac{1}{4}\frac{\cos\left(\frac{3\pi}{2}u\right)}{1-u^2},$$

with initial data

$$u = \operatorname{th}\frac{4x}{\varepsilon}, \qquad v = 2\operatorname{th}\frac{4x}{\varepsilon}. \tag{5}$$

By direct substitution we verify that the system of functions (5) is a solution of this problem.

On the other hand, consider the system of equations

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\cos\frac{3\pi}{2}u\right) &= \varepsilon\frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial v}{\partial t} - \frac{\partial v^2}{\partial x} &= \varepsilon\frac{\partial^2 v}{\partial x^2}. \end{aligned} \tag{6}$$

with the same initial data (5). System (6) consists of two independent equations. Therefore, using the results obtained in papers (^{1}, ^{2}), one can show that problem (6), (5), for sufficiently small ε, has a solution close to the following:

$$u = \begin{cases} \operatorname{sign} x, & \text{for } \left|\frac{x}{t}\right| > a, \\ \frac{2}{3\pi}\arcsin\left(\frac{2}{3\pi}\frac{x}{t}\right), & \text{for } \left|\frac{x}{t}\right| < a; \end{cases} \qquad v = 2\operatorname{sign} x, \tag{7}$$

where a is the smallest positive root of the equation

$$a^2\left(\left(\frac{2}{3\pi}\right)^2 + \left(1 - \frac{2}{3\pi}\arcsin\frac{2a}{3\pi}\right)^2\right) = 1.$$

Systems (4) and (6) are evolutionary; they approximate the same system of hyperbolic equations

$$\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left(\cos\frac{3\pi}{2}u\right) = 0,$$

$$\frac{\partial v}{\partial t} - \frac{\partial v^2}{\partial x} = 0$$

and differ only in the viscosity matrices $B(u)$. Nevertheless, their limiting solutions (as $\varepsilon \to 0$) clearly do not coincide. One is

$$u = \operatorname{sign} x, \qquad v = 2 \operatorname{sign} x,$$

and the other is (7).

In constructing the viscosity matrix in system (4), ideas contained in the work of S. K. Godunov (^{3}) were used.

I take this opportunity to express my gratitude to I. M. Gelfand for valuable discussions.

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8 VII 1960

CITED LITERATURE

[1] I. M. Gelfand, *Uspekhi Mat. Nauk*, **14**, no. 2 (86) (1959).
[2] A. S. Kalashnikov, *DAN*, **127**, no. 1 (1959).
[3] S. K. Godunov, *DAN*, **134**, no. 6 (1960).

Note: Figure translations are in progress. See original paper for figures.