SOME INEQUALITIES FOR POLYNOMIALS OF A COMPLEX VARIABLE

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Abstract

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Let $Q_n(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$ be a polynomial of degree n. S. N. Bernstein ([1]) proved the following assertion:

If $|Q_n(z)| \leq 1$ for $|z| \leq 1$, then the inequality $|Q_n'(z)| \leq n$ holds for $|z| = 1$.

Let

$$\|Q_n(re^{i\varphi})\|_p = \left\{ \int_0^{2\pi} |Q_n(re^{i\varphi})|^p \, d\varphi \right\}^{1/p}.$$

It is not difficult to show (for example, see ([5])) that for any $p \geq 1$ the inequalities

$$\|Q_n(Re^{i\varphi})\|_p \leq R^n \|Q_n(e^{i\varphi})\|_p \quad (R > 1); \tag{1}$$

$$\|Q_n(\rho e^{i\varphi})\|_p \geq \rho^n \|Q_n(e^{i\varphi})\|_p \quad (\rho < 1). \tag{2}$$

hold.

For polynomials $Q_n(z)$ having no zeros inside the unit disk $|z| < 1$, de Bruijn ([4]) proved the following assertion:

Theorem A. If the polynomial $Q_n(z)$ of degree n has no zeros in $|z| < 1$, then for any $p \geq 1$ the inequality

$$\|Q_n'(e^{i\varphi})\|_p \leq \frac{n}{2} \left[\frac{\sqrt{\pi}\, \Gamma\left(\frac{1}{2}p + 1\right)}{\Gamma\left(\frac{1}{2}(p+1)\right)} \right]^{1/p} \|Q_n(e^{i\varphi})\|_p. \tag{3}$$

In relation (3), the equality sign is attained only for polynomials of the form $Q_n(z) = \lambda + \mu z^n$, where $|\lambda| = |\mu|$.

In this note we refine inequalities (1) and (2) for polynomials $Q_n(z)$ of degree n that have no zeros in $|z| < 1$.

Theorem 1. If $Q_n(z)$ has no zeros in $|z| < 1$, then for $p \geq 1$ the inequality

$$\|Q_n(Re^{i\varphi})\|_p < \left\{ \frac{1}{2} \left[\frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}p+1\right)}{\Gamma\left(\frac{1}{2}(p+1)\right)} \right]^{1/p} (R^n - 1) + 1 \right\} \|Q_n(e^{i\varphi})\|_p \qquad (4)$$

holds for any $R > 1$.

Proof. Suppose first that $Q_n(z) \neq \lambda + \mu z^n$, where $|\lambda| = |\mu|$. From the obvious inequality

$$|Q_n(Re^{i\varphi})| \leq \int_1^R |Q_n'(re^{i\varphi})|\, dr + |Q_n(e^{i\varphi})|,$$

where φ $(0 \leqslant \varphi \leqslant 2\pi)$ is any number and $R > 1$, for any $\rho \geqslant 1$, the inequality follows:

$$\|Q_n(Re^{i\varphi})\|_p \leqslant \int_1^R \|Q_n'(re^{i\varphi})\|_p\, dr + \|Q_n(e^{i\varphi})\|_p.$$

Taking (1) into account, the last inequality may be written in the form

$$\|Q_n(Re^{i\varphi})\|_p \leqslant \|Q_n'(e^{i\varphi})\|_p \int_1^R r^{n-1}\, dr + \|Q_n(e^{i\varphi})\|_p.$$

Hence, by Theorem A it follows that

$$\|Q_n(Re^{i\varphi})\|_p < \left\{ \frac{n}{2} \left[\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}p+1\right)}{\Gamma\left(\frac{1}{2}(p+1)\right)} \right]^{1/p} \frac{R^n - 1}{n} + 1 \right\} \|Q_n(e^{i\varphi})\|_p,$$

i.e. (4) is valid.

It remains to verify the validity of (4) for polynomials of the form $Q_n(z) = \lambda + \mu z^n$, where $|\lambda| = |\mu|$. This follows from the inequality

$$\|\lambda + \mu e^{i\varphi} R^n\|_p < (R^n - 1)|\mu|(2\pi)^{1/p} + \|\lambda + \mu e^{i\varphi n}\|_p =$$

$$= \left\{ \frac{1}{2} \left[\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}p+1\right)}{\Gamma\left(\frac{1}{2}(p+1)\right)} \right]^{1/p} (R^n - 1) + 1 \right\} \|\lambda + \mu z^n\|_p.$$

In the case $p = \infty$, Theorem 1 was proved by Ankeny and Rivlin (3), and in the case $p = 1$, by Rahman (5).

Theorem 2. *If $Q_n(z)$ has no zeros in $|z| < 1$, then there exists a positive number δ such that, for $(1 - \delta) < \rho < 1$, the inequality*

$$\|Q_n'(\rho e^{i\varphi})\|_p > \left\{ 1 - \frac{1}{2} \left[\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}p+1\right)}{\Gamma\left(\frac{1}{2}(p+1)\right)} \right]^{1/p} (1 - \rho^n) \right\} \|Q_n(e^{i\varphi})\|_p, \qquad (5)$$

holds, where $\rho \geqslant 1$ is any number.

Proof. The validity of inequality (5) for polynomials $Q_n(z) = \lambda + \mu z^n$, where $|\lambda| = |\mu|$, having no zeros in $|z| < 1$, is verified directly. Indeed,

$$\|\lambda + \mu\rho^n e^{i\varphi n}\|_p > \|\lambda + \mu e^{i\varphi n}\|_p - (1-\rho^n)|\mu|(2\pi)^{1/p} =$$

$$= \left\{1 - \frac{1}{2}\left[\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}p+1\right)}{\Gamma\left(\frac{1}{2}(p+1)\right)}\right]^{1/p}(1-\rho^n)\right\}\|\lambda + \mu e^{i\varphi n}\|_p,$$

i.e. (5) is valid.

Now we prove the theorems for polynomials $Q_n(z) \neq \lambda + \mu z^n$ of degree n, where $|\lambda| = |\mu|$, having no zeros in $|z| < 1$. For the proof suppose the contrary, i.e. that (5) does not hold. This means that there exists a polynomial $Q_n(z) \neq \lambda + \mu z^n$ ($|\lambda| = |\mu|$) of degree n, having no zeros in $|z| < 1$, and a sequence of values $1 - \delta < \rho_m < 1$ $(m = 1, 2, ...)$ with $\lim\limits_{m\to\infty} \rho_m = 1$, such that

$$\|Q_n(\rho_m e^{i\varphi})\|_p \leqslant \left\{1 - \frac{1}{2}\left[\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}p+1\right)}{\Gamma\left(\frac{1}{2}(p+1)\right)}\right]^{1/p}(1-\rho_m^n)\right\}\|Q_n(e^{i\varphi})\|_p. \quad (6)$$

Then

$$\|Q_n'(e^{i\varphi})\|_p = \left\|\lim_{m\to\infty}\frac{Q_n(e^{i\varphi}) - Q_n(\rho_m e^{i\varphi})}{e^{i\varphi} - \rho_m e^{i\varphi}}\right\|_p \geqslant$$

$$\geqslant \lim_{m\to\infty}\frac{1}{1-\rho_m}\left(\|Q_n(e^{i\varphi})\|_p - \|Q_n(\rho_m e^{i\varphi})\|_p\right).$$

Hence, by virtue of (6), we have

$$\left\|Q_n'\left(e^{i\varphi}\right)\right\|_p \geq \frac{1}{2}\left[\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}p+1\right)}{\Gamma\left(\frac{1}{2}(p+1)\right)}\right]^{1/p}\left\|Q_n\left(e^{i\varphi}\right)\right\|_p \lim_{m\to\infty}\frac{1-\rho_m^n}{1-\rho_m}$$
$$= \frac{n}{2}\left[\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2}p+1\right)}{\Gamma\left(\frac{1}{2}(p+1)\right)}\right]^{1/p}\left\|Q_n\left(e^{i\varphi}\right)\right\|_p,$$

which contradicts Theorem A. Thus, inequality (5) is proved also for all $Q_n(z) \neq \lambda + \mu z^n$, where $|\lambda| = |\mu|$. For $p = \infty$, it follows from Theorem 2 that if $Q_n(z)$ has no zeros in $|z| < 1$ and $\max_{|z|=1}|Q_n(z)| = 1$, then there exists a positive number $\delta > 0$ such that

$$\max_{|z|=\rho}|Q_n(z)| \geq \frac{1+\rho^n}{2}$$

for $1 - \delta < \rho < 1$.

For $p = 1$, one theorem of Rahman follows from Theorem 2 ([5]).

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Note: Figure translations are in progress. See original paper for figures.