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Abstract

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SOME UNIQUENESS THEOREMS FOR FUNCTIONS SUBHARMONIC AND MEROMORPHIC IN THE UNIT DISK

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We shall consider systems of disks C_i lying inside the disk $|z| < 1$ and having a finite sum of Euclidean radii. The set of points belonging to such a system of disks will be denoted by C. We shall give a lower estimate outside the set C for functions subharmonic in the unit disk, and also for functions represented as the difference of two subharmonic functions. On the basis of these estimates, we shall establish certain uniqueness theorems for functions meromorphic in the disk $|z| < 1$.

Let $U(z)$ be a subharmonic function in the disk $|z| < 1$. Put

$$U^+(z) = \begin{cases} U(z), & \text{if } U(z) \geq 0, \\ 0, & \text{if } U(z) < 0, \end{cases}$$

and consider the function

$$\varphi(r, U) = \frac{1}{2\pi} \int_0^{2\pi} U^+(re^{i\theta})\, d\theta \qquad (0 \leq r < 1)$$

(sometimes we shall write, for short, $\varphi(r)$). Let $\sigma(r)$ be such a positive monotone function that

$$\int_0^1 \varphi(r)\sigma(r)\, dr < M. \tag{1}$$

Put

$$\chi(t) = \int_t^1 \sigma(t)\, dt, \qquad \mu(t) = \int_t^1 \chi(\tau)\, d\tau.$$

From the integral representation of F. Riesz ([2]) it follows that to every subharmonic function in the disk $|z| < 1$ there corresponds a certain distribution* of nonnegative masses $m(z)$. Denote

$$n(t) = \iint_{|z| \leq t} dm(z).$$

With the aid of Jensen' s theorem for subharmonic functions it is easy

* To obtain this distribution, one should represent the function $U(z)$ according to Riesz in the disk $|z| \leq r < 1$ and observe that the distribution obtained does not change as $r \uparrow 1$.

show that

$$\int_0^1 \mu(t)\, dn(t) < \infty. \tag{2}$$

This condition on the masses is an analogue of the Blaschke condition for the zeros of bounded holomorphic functions in the disk $|z| < 1$.

Theorem 1. *For an arbitrary subharmonic function $U(z)$ in the unit disk with characteristic $\varphi(r)$, satisfying condition (1), there exists a set C such that*

$$U(z) > -\frac{N_0}{\mu(|z| + \theta(1 - |z|))} \quad (\bar{z} \in C,\ 0 < \theta < 1). \tag{3}$$

The proof of this theorem is carried out by means of Hayman' s method ([1]), proposed by him for obtaining estimates of this kind as applied to bounded subharmonic functions in a half-plane.

Remark. Choosing $\sigma(t) = \varphi^{-1}(t)(1-t)^{\varepsilon-1}$, we obtain $\mu(t) > C_\varepsilon \varphi^{-1}(t)(1-t)^{1+\varepsilon}$ and shall have for $U(z)$ an estimate directly in terms of its characteristic $\varphi(r, U)$:

$$U(z) > -\frac{N_{\theta,\varepsilon}\varphi(|z| + \theta(1 - |z|))}{(1 - |z|)^{1+\varepsilon}} \quad (\bar{z} \in C),$$

Let us consider some important special cases of Theorem 1. Suppose the function $\varphi(r)$ has finite order $\lambda \geqslant 0$, i.e.

$$\overline{\lim_{r \to 1}} \frac{\log \varphi(r)}{\log\left(\frac{1}{1-r}\right)} = \lambda.$$

Then λ is the critical exponent of the integral

$$\int_0^1 \varphi(r)(1-r)^{\mu-1}\,dr, \tag{4}$$

i.e. the exact lower bound of those μ for which integral (4) converges. We shall distinguish the convergence class and the divergence class according as the integral converges or diverges for $\mu = \lambda$. If the characteristic $\varphi(r)$ belongs to the convergence class, then $\sigma(r) = (1-r)^{\lambda-1}$, and for $U(z)$ the relation

$$U(z) > -\frac{N}{(1-|z|)^{\lambda+1}} \quad (\bar{z} \in C).$$

is valid. If $\varphi(r)$ belongs to the divergence class, then one may put $\sigma(r) = (1-r)^{\lambda+\varepsilon-1}$, and for $U(z)$ the inequality

$$U(z) > -\frac{N_\varepsilon}{(1-|z|)^{\lambda+\varepsilon+1}} \quad (\bar{z} \in C).$$

will be valid.

For subharmonic functions $U(z)$ with bounded characteristic $\varphi(r, U)$, it turns out to be possible to put $\mu(|z|) = 1-|z|$ and obtain the sharper result:

$$U(z) > -\frac{N}{1-|z|} \quad (\bar{z} \in C).$$

If $U(z) \leqslant 0$ $(|z| < 1)$ and $\int_0^{2\pi} U(re^{i\theta})\,d\theta \to 0$ $(r \to 1)$, then for every $\varepsilon > 0$

$$U(z) > -\frac{\varepsilon}{1-|z|} \quad (\bar{z} \in C,\ 1-|z| < \delta(\varepsilon)). \tag{5}$$

The estimate (3) is sharp in the class of all possible subharmonic functions in the disk $|z| < 1$. Thus, for subharmonic functions whose characteristic has order λ, for any $\varepsilon > 0$ we have

$$U(z) > -\frac{N_\varepsilon}{(1-|z|)^{1+\lambda+\varepsilon}} \quad (z \bar{\in} C). \tag{6}$$

It is impossible to replace λ in (6) by a smaller quantity $\lambda_1 < \lambda$, as is shown by the example

$$U(z) = \mathrm{Re}\left(\frac{1+z}{1-z}\right)^{\lambda+1}.$$

Here $\varphi(r, U)$ has order λ, and $U(z) > -N/(1-|z|)^{1+\lambda}$.

Let us consider the difference $U(z) = U_1(z) - U_2(z)$ of two subharmonic functions in the disk $|z| < 1$, and introduce the characteristic function $T(r, U) = \varphi(r, U) + N(r, U_2)$, where $\varphi(r, U)$ is defined above, and

$$N(r, U_2) = \int_0^r \frac{n(t, U_2)}{t}\, dt$$

$*$. Let $\sigma(r)$ again be such a monotone function that

$$\int_0^1 T(r)\sigma(r)\, dr < M. \tag{7}$$

Put, as before,

$$\chi(t) = \int_t^1 \sigma(\tau)\, d\tau, \qquad \mu(t) = \int_t^1 \chi(\tau)\, d\tau. \tag{8}$$

Then the following holds.

Theorem 2. *If the characteristic of the difference $U(z)$ of two subharmonic functions satisfies condition (7), then*

$$U(z) > -\frac{N_\theta}{\mu(|z| + \theta(1 - |z|))} \quad (z\bar{\in}C,\ 0 < \theta < 1). \tag{9}$$

Let $f(z)$ be meromorphic in the disk $|z| < 1$. Then $\ln|f(z)|$ can be represented in the form of a difference of two subharmonic functions, and the characteristic of the difference introduced by us will, obviously, coincide with the Nevanlinna characteristic $T(r, f)$. Defining for $U(z) = \ln|f(z)|$ the function $\mu(|z|)$ as indicated above, we can write inequality (9) in the form

$$\ln|f(z)| > -\frac{N_\theta}{\mu(|z| + \theta(1 - |z|))} \quad (z\bar{\in}C,\ 0 < \theta < 1).$$

By $\{z_k\}_1^\infty$ we shall mean a sequence of points with moduli $|z_k| = r_k$ satisfying the condition $r_k \to 1$. We shall call such a sequence an S_μ-sequence if the points cannot be enclosed in any set C such that $\sum_j \mu(R_j) < \infty$, where R_j are the moduli of the centers of the disks C_j.

Theorem 3. *If the characteristic $T(r, f)$ of a function $f(z)$, meromorphic in the disk $|z| < 1$, satisfies condition (7), and the points $\{z_k\}$ form an S_μ-sequence with the function $\mu(|z|)$ defined by equalities (8), then from the relation*

$$\lim_{k\to\infty} \mu(|z_k| + \theta(1 - |z_k|)) \ln|f(z_k)| = -\infty$$

for some θ $(0 < \theta < 1)$, it follows that the function $f(z)$ is identically equal to zero.

* Without loss of generality, one may assume that some neighborhood of zero is free of masses.

Theorem 3 gives a partial answer to the question of how rapidly a meromorphic function in the disk $|z| < 1$ can decrease along a sequence of points approaching the boundary of the disk.

A related question is considered in the works of A. L. Shaginyan ([3,4]). However, in his case the approach to the boundary occurs not along a sequence of points, but along a set of positive linear measure lying on a Jordan curve going to the boundary of the disk $|z| < 1$.*

I shall give several examples of sufficient conditions for the sequence $\{z_k\}_1^\infty$ to be an S_μ-sequence. Such conditions are the condition

$$\sum_k \mu(|z_k|) = \infty \tag{A}$$

in combination with any one of the noncondensation conditions listed below**

$$d(z_n, z_m) \geq \delta |n - m| \qquad (\delta > 0); \tag{B_1}$$

here $d(z_n, z_m)$ is the non-Euclidean distance between the points z_n and z_m,

$$\frac{|z_n - z_m|}{\sqrt{(1 - |z_n|)(1 - |z_m|)}} \geq \delta \sqrt{|n - m|} \qquad (\delta > 0); \tag{B_2}$$

$$\left[\frac{1}{\mu(|z_{k+1}|)} - \frac{1}{\mu(|z_k|)} \right] \frac{\mu(|z_k|)}{(1 - |z_k|)(-\mu'(|z_k|))} \geq \delta > 0. \tag{B_3}$$

In (B_3) it is assumed that the points z_k have been renumbered in increasing order of their moduli.

Let us note that the noncondensation conditions can be chosen in various ways. However, without some condition of this kind the assertion of the theorem is false. Indeed, let us take a meromorphic function with zeros $\{a_k\}$, which (see (2)) must satisfy the condition $\sum \mu(|a_k|) < \infty$, and form the sequence $\{z_k\}_1^\infty$ from the zeros a_k and a sufficiently large number of points from small neighborhoods of the zeros so that $\sum_k \mu(|z_k|) = \infty$, while the sequence $|f(z_k)|$ decreases arbitrarily rapidly.

I take this opportunity to express my sincere gratitude to Prof. B. Ya. Levin for suggesting the topic and for help in the work.

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CITED LITERATURE

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4. A. L. Shaginyan, *Izv. AN Arm SSR, ser. fiz.-matem. nauk*, **12**, No. 1 (1959).

5. I. V. Ushakova, *DAN*, **130**, No. 1 (1960).

6. I. V. Ushakova, Abstracts of Reports, Fifth All-Union Conference on Function Theory, Erevan, 1960.

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* In a report at the Fifth All-Union Conference on Function Theory, A. L. Shaginyan noted that an estimate of the rate of decrease of the function $f(z)$ along a discrete sequence of points z_k can be obtained by reduction to the case studied in ([3],[4]); however, for such a reduction it is required that the points z_k be situated very densely on the curve. Here we do not make such an assumption.

I also note that S. Ya. Khavinson, in his report, presented general considerations giving a new approach to the question of how rapidly a meromorphic function of bounded type can decrease along a sequence of points approaching the boundary.

** If one sets $\mu(t) = 1 - t$, which corresponds, as was indicated, to meromorphic functions of bounded type, then we obtain a uniqueness theorem including the uniqueness theorem from ([5],[6]).

Note: Figure translations are in progress. See original paper for figures.