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Abstract

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MATHEMATICS

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ON THE QUESTION OF REFINING THE ESTIMATE OF THE SPECTRAL FUNCTION OF THE LAPLACE OPERATOR

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We introduce the following notation. Let g be an N-dimensional domain for which the eigenvalue problem for the Laplace operator is solvable under a homogeneous boundary condition of any of the three kinds; P is an arbitrary fixed point of the domain g; Q is any point of the domain g; $u_i(x)$ is the i-th eigenfunction of the problem under consideration; λ_i is the i-th eigenvalue. We shall call the function

$$\vartheta(P,Q) = \sum_{\sqrt{\lambda_i}<\mu} u_i(P)u_i(Q)$$

the spectral function of the Laplace operator.

As a special case of the domain g, an arbitrary N-dimensional rectangular parallelepiped is considered.

Theorem 1. For the spectral function $\vartheta(P,Q)$ of the Laplace operator under a homogeneous boundary condition of the 1st or 2nd kind for an N-dimensional rectangular parallelepiped, in the case when $P \neq Q$, the following asymptotic estimate holds:

$$\vartheta(P,Q) = \sum_{\sqrt{\lambda_i}<\mu} u_i(P)u_i(Q) = O(\mu^{N-\nu}), \tag{1}$$

where

$$\nu = \frac{2N}{N+1} \quad \text{for } N < 8; \qquad \nu = 2 \quad \text{for } N \geq 8. \tag{2}$$

Estimate (1) is uniform with respect to P and Q in the entire N-dimensional rectangular parallelepiped from which an arbitrarily small neighborhood of the point $Q = P$ has been removed.

Proof. Without restricting the generality of the proof, consider an N-dimensional parallelepiped with sides $[0, a_n]$ $(n = 1, 2, \ldots, N)$. We shall carry out the proof for a homogeneous boundary condition of the 1st kind.

The eigenfunctions of the problem

$$\Delta u + \lambda u = 0, \quad u|_\Gamma = 0$$

in this case are the functions:

$$u_{k_1 \dots k_N}(x_1, \dots, x_N) = \prod_{n=1}^{N} \left(\sqrt{\frac{2}{a_n}} \right) \sin \frac{\pi k_n}{a_n} x_n,$$

$$0 \le x_n \le a_n, \quad n = 1, 2, \dots, N,$$

and the eigenvalues are the numbers:

$$\lambda_{k_1, \dots, k_N} = \pi^2 \sum_{n=1}^{N} \frac{k_n^2}{a_n^2}.$$

Let the points P and Q, belonging to our parallelepiped, be given respectively by the coordinates $x_1, x_2, \dots, x_N$; $\xi_1, \xi_2, \dots, \xi_N$. The spectral function takes the form

$$\vartheta(x_1, \dots, x_N, \xi_1, \dots, \xi_N) = \sum_{\sqrt{\lambda_i} < \mu} \prod_{n=1}^{N} \left(\frac{2}{a_n} \right) \sin \frac{k_n \pi}{a_n} x_n \cdot \sin \frac{k_n \pi}{a_n} \xi_n. \tag{3}$$

Substituting in (3) the values of the sines expressed in terms of exponentials, and carrying out the multiplication, we split the sum (3) into 4^N sums of the form

$$S'(\mu) = C \sum_{\sqrt{\lambda_i} < \mu} e^{2\pi i (k_1 \gamma_1 + \dots + k_N \gamma_N)}. \tag{4}$$

Here

$$C = \pm \prod_{n=1}^{N} \frac{1}{2a_n},$$

and γ_n $(n = 1, 2, \dots, N)$ is one of the numbers

$$\frac{x_n + \xi_n}{2a_n}; \quad -\frac{x_n + \xi_n}{2a_n}; \quad \frac{x_n - \xi_n}{2a_n}; \quad -\frac{x_n - \xi_n}{2a_n}.$$

It follows from the definition of γ_n that, for $P \ne Q$, among the numbers γ_n, $n = 1, 2, \dots, N$, at least one will be nonintegral. In the sum (4) the summation is over the coordinates of the integer points of the domain

$$\pi^2 \sum_{n=1}^{N} \frac{k_n^2}{a_n^2} < \mu^2, \qquad k_n \geq 1 \quad (n = 1, 2, \dots, N).$$

Since

$$\pi^2 \sum_{n=1}^{N} \frac{k_n^2}{a_n^2}$$

is a positive-definite quadratic form and $P \neq Q$, for sums of the form (4) the asymptotic estimate obtained by Landau–Hardy [(1)] is valid:

$$\sum_{\sqrt{\lambda_i}<\mu} e^{2\pi i(k_1\gamma_1+\cdots+k_N\gamma_N)} = O\left(\mu^{N-\nu}\right), \tag{5}$$

where $\nu = \dfrac{2N}{N+1}$ for $N < 8$ and $\nu = 2$ for $N \geq 8$. Using the estimate (5), we obtain the desired estimate for the spectral function. It is clear that this estimate is uniform throughout the N-dimensional rectangular parallelepiped from which an arbitrarily small neighborhood of the point $Q = P$ has been removed. The theorem is proved.

For the spectral function of an arbitrary domain g, in the case when $P \neq Q$, the asymptotic estimate [(2,3)] had previously been obtained

$$\sum_{\sqrt{\lambda_i}<\mu} u_i(P)\, u_i(Q) = O\left(\mu^{N-1}\right).$$

Thus, for the special case we have succeeded in somewhat sharpening the asymptotic estimate of the spectral function of the Laplace operator. It is easy to show that, for $P \neq Q$, the asymptotic estimate of the spectral function cannot have order in μ lower than

$$O\left(\mu^{\frac{N-1}{2}}\right).$$

The question of whether estimate (1) can be improved to this order in μ remains open.

Let us apply estimate (1) to the proof of an expansion theorem in a conditionally convergent Fourier series in the eigenfunctions of the Laplace operator for functions possessing a singularity. It is known [(5)] that if a function has, at an interior fixed point P of the domain, a singularity of the type $\ln r_{PQ}$ or $1/r_{PQ}^{\alpha}$ ($\alpha > 0$), then one cannot expect absolute convergence of its Fourier series at any interior point of the domain, even if, after subtraction of the singularity, this function satisfies arbitrarily high smoothness requirements; consequently, one can speak only of expansion in a conditionally convergent Fourier series.

Theorem 2. If a function of N variables $v(P,Q)$, given in an arbitrary N-dimensional rectangular parallelepiped, has at an interior point P of this parallelepiped a singularity $1/r_{PQ}^{\alpha}$ $(0 < \alpha < \nu)$ and, after subtraction of this singularity, satisfies the usual conditions of expandability*, then this function can be expanded in a Fourier series in the eigenfunctions

* By the term *usual conditions of expandability* we mean the conditions established in the main theorem of the work [(6)].

solutions of the equation $\Delta u + \lambda u = 0$ in this parallelepiped with a homogeneous boundary condition of the 1st or 2nd kind, and the indicated series converges, when summed in the order of increasing eigenvalues, uniformly in the whole N-dimensional rectangular parallelepiped, from which an arbitrarily small neighborhood of the singular point has been removed.

Remark. In what follows we shall assume that in all conditionally convergent series considered by us the summation is carried out in the order of increasing eigenvalues.

Proof. Since $\alpha < \nu$, there exists an $\varepsilon > 0$ such that $\alpha = \nu - 2\varepsilon$. It is known ([5]) that, in the case of an N-dimensional rectangular parallelepiped, the kernel of fractional order

$$K_{\frac{N}{2}-\frac{\nu}{2}+\varepsilon}(P,Q),$$

which possesses the above-mentioned singularity and has as its Fourier series the bilinear series

$$\sum_{i=1}^{\infty} \frac{u_i(P)u_i(Q)}{\lambda_i^{\frac{N}{2}-\frac{\nu}{2}+\varepsilon}}, \tag{6}$$

after separation of the singularity, is an arbitrarily smooth function of Q in the whole closed parallelepiped. Obviously, there exists a constant C such that the function

$$f(P,Q) = C\nu(P,Q) - K_{\frac{N}{2}-\frac{\nu}{2}+\varepsilon}(P,Q)$$

satisfies, throughout the parallelepiped, the conditions for expandability in a conditionally convergent Fourier series. Hence it follows that it is sufficient to prove the theorem for the function

$$K_{\frac{N}{2}-\frac{\nu}{2}+\varepsilon}(P,Q) = \sum_{i=1}^{\infty} \frac{u_i(P)u_i(Q)}{\lambda_i^{\frac{N}{2}-\frac{\nu}{2}+\varepsilon}}.$$

Let $\varepsilon = \varepsilon_1 + \varepsilon_2$ $(\varepsilon_1 > 0,\ \varepsilon_2 > 0)$, $\alpha_1 = N - \nu$. We have

$$K_{\frac{\alpha_1}{2}+\varepsilon}(P,Q) = \sum_{i=1}^{\infty} \frac{u_i(P)u_i(Q)}{\lambda_i^{\frac{\alpha_1}{2}+\varepsilon_1}} \left\{\frac{1}{\lambda_i^{\varepsilon_2}}\right\}.$$

The sequence $\left\{\frac{1}{\lambda_i^{\frac{\varepsilon}{2}}}\right\}$ tends monotonically to zero. By Abel' s test it is enough to prove the uniform boundedness of the partial sums of the series

$$\sum_{i=1}^{\infty}\frac{u_i(P)u_i(Q)}{\lambda_i^{\frac{\alpha_1}{2}+\varepsilon_1}} \tag{7}$$

under the condition $\rho(P,Q)\geq\delta>0$. Without loss of generality we assume $\lambda_1\geq 1$.

We apply Abel' s transformation:

$$\sum_{i=1}^{n}\frac{u_i(P)u_i(Q)}{\lambda_i^{\frac{\alpha_1}{2}+\varepsilon_1}}=\sum_{k=1}^{n-1}\left(\sum_{\sqrt{\lambda_i}<\sqrt{\lambda_k}}u_i(P)u_i(Q)\right)\left[\frac{1}{\lambda_k^{\frac{\alpha_1}{2}+\varepsilon_1}}-\frac{1}{\lambda_{k+1}^{\frac{\alpha_1}{2}+\varepsilon_1}}\right]+\left[\sum_{\sqrt{\lambda_i}<\sqrt{\lambda_n}}u_i(P)u_i(Q)\right]\frac{1}{\lambda_n^{\frac{\alpha_1}{2}+\varepsilon_1}}. \tag{8}$$

It is easy to show that from estimate (1) there follows the uniform boundedness of the right-hand side of equality (8), and here the order of summation plays an essential role. Thus the uniform convergence of series (6) has been proved in the entire N-dimensional rectangular parallelepiped, except for an arbitrarily small neighborhood of the singular point.

The convergence of series (6) precisely to the kernel $K_{\frac{N}{2}-\frac{\nu}{2}+\varepsilon}(P,Q)$ follows from the completeness and closedness of the system of eigenfunctions under consideration in the spaces L_p $(p\geq 1)$.

Theorem 2 strengthens, for the given particular type of domain, a theorem of V. A. Il' in, in which, for an arbitrary domain, uniform convergence is established for the Fourier series of functions having a singularity of the type $\ln r_{PQ}$ or $1/r_{PQ}^{\alpha}$ $(0<\alpha<1)$ in any strictly interior subdomain g' of the basic domain g, from which an arbitrarily small neighborhood of the singular point has been removed.

Corollary 1. From Theorem 2 it follows that for the case under consideration of an N-dimensional rectangular parallelepiped, when $P\neq Q$, the formula

$$K_{\frac{N}{2}-\frac{\alpha}{2}}(P,Q)=\sum_{i=1}^{\infty}\frac{u_i(P)u_i(Q)}{\lambda_i^{\frac{N}{2}-\frac{\alpha}{2}}}; \tag{9}$$

holds for $0<\alpha<\nu$; ν satisfies conditions (2). For the present case, (9) replaces the known theorem of Mercer, extending it to kernels possessing the singularity indicated above. It should be noted that, for $N=2$, the function $K_{\frac{N}{2}-\frac{\alpha}{2}}(P,Q)$ may have a singularity of order $1/r^{\frac{4}{3}-\delta}$ $(\delta>0)$ and may fail to be square integrable. In this case the series (9) converges uniformly everywhere in the domain, except for an arbitrarily small neighborhood of the singular point, but does not converge in this domain in the mean.

Corollary 2. For $\alpha = 1$ and $N = 3$ we obtain the following representation of the Green's function of the Laplace operator for the 1st and 2nd boundary-value problems in a three-dimensional rectangular parallelepiped:

$$K_1(P, Q) = \sum_{i=1}^{\infty} \frac{u_i(P)u_i(Q)}{\lambda_i}. \tag{10}$$

This formula is valid for any two points P and Q of the indicated parallelepiped such that $P \neq Q$, and gives an answer to the question posed in the book of R. Courant ((7), p. 324).

Obviously, the series (9) and (10) converge uniformly in the entire parallelepiped from which an arbitrarily small neighborhood of the point $Q = P$ has been removed.

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Note: Figure translations are in progress. See original paper for figures.