MATHEMATICS

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Abstract

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ON AN INTEGRAL INEQUALITY AND THE CORRESPONDING EMBEDDING THEOREM

(Presented by Academician S. L. Sobolev on 29 XI 1960)

Let R_n be n-dimensional Euclidean space. The distance between the points $x, y \in R_n$ will be denoted by $r(x, y)$ $(r(x, 0) = r(x))$.

1. In ([1]) the inequality

$$\iint\limits_{R_s R_n} \frac{f(x)g(y)}{r^{-k}(x)[r(x,y)]^{n/p'+s/q-h+k} r^h(y)}\, dx^{(n)} dy^{(s)} \leq C_1 \|f\|_{L_p(R_n)} \|g\|_{L_{q'}(R_s)}, \quad (1)$$

was proved, where $1 < p \leq q;\ s \leq n;\ -n/p' < k < h < s/q$, and the constant C_1 depends only on n, p, k, s, q, h. In the case $n = 1$, inequality (1) was established by Hardy and Littlewood ([2], p. 359, item 401).

As Hardy, Littlewood, and Pólya showed ([2], p. 346, item 382) for $n = 1$; S. L. Sobolev ([3]) for $s = n > 1$; V. P. Il' in ([4]) for $s < n \neq 1$, inequality (1) is valid for $h = k = 0$ $(p < q)$. We shall denote inequality (1) for $h = k = 0$ by $(1')$, and the constant of inequality $(1')$ by c_1'.

Let R_t be a subspace of R_n of dimension $t < n$. By R_{n-t} we shall denote the orthogonal complement to R_t in R_n. The following inequality is valid, generalizing inequality (1):

Theorem 1. Let $R_s \subseteq R_n$ $(s \leq n)$ and $R_\sigma = R_t \cap R_s$ $(\sigma > 0)$. Suppose that $R_{s-\sigma} \subseteq R_{n-t}$ $(0 < s - \sigma \leq n - t)$.

If the numbers p, q, k, h satisfy the conditions

$$1 < p < q; \qquad -\frac{n-t}{p'} < k < h < \frac{s-\sigma}{q},$$

then the inequality

$$I = \iint\limits_{R_s R_n} \frac{f(x)g(y)}{r_{n-t}^{-k}(x)[r(x,y)]^{n/p'+s/q-h+k} r_{s-\sigma}^{h}(y)}\, dx^{(n)} dy^{(s)} \le c_2 \|f\|_{L_p(R_n)} \|g\|_{L_{q'}(R_s)}, \tag{2}$$

is valid, where $r_{n-t}(x)$ is the distance from $x \in R_n$ to R_t; $r_{s-\sigma}(y)$ is the distance from $y \in R_s$ to R_σ; $c_2 = c_1 c_1'$ is a constant depending only on $n, p, k, t, s, q, h, \sigma$.

The proof of inequality (2) is based on inequalities (1), (1′) and is carried out with the aid of a device of Plessner ([5]).

Indeed, without loss of generality one may assume that

$$\begin{aligned}
R_t &: \ x_{t+1} = x_{t+2} = \dots = x_n = 0; \\
R_{n-t} &: \ x_1 = x_2 = \dots = x_t = 0; \\
R_s &: \ x_{\sigma+1} = x_{\sigma+2} = \dots = x_t = \dots = x_{n-s+\sigma} = 0; \\
R_\sigma &: \ x_{\sigma+1} = x_{\sigma+2} = \dots = x_n = 0; \\
R_{s-\sigma} &: \ x_1 = x_2 = \dots = x_t = \dots = x_{n-s+\sigma} = 0.
\end{aligned}$$

Then for any $x \in R_n$ and $y \in R_s$ the estimate

$$r(x,y) \ge \tag{3}$$

$$\ge \begin{cases} \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_\sigma - y_\sigma)^2 + x_{\sigma+1}^2 + x_{\sigma+2}^2 + \dots + x_t^2} \equiv r_t(x,y), \\ \sqrt{x_{t+1}^2 + x_{t+2}^2 + \dots + x_{n-s+\sigma}^2 + (x_{n-s+\sigma+1} - y_{n-s+\sigma+1})^2 + \dots + (x_n - y_n)^2} \equiv r_{n-t}(x,y). \end{cases}$$

With the aid of estimate (3) we obtain

$$I \le \int_{R_{s-\sigma}} \int_{R_{n-t}} \frac{1}{r_{n-t}^{-k}(x)\, [r_{n-t}(x,y)]^{(n-t)/p'+(s-\sigma)/q-h+k}\, r_{s-\sigma}^{h}(y)} \times$$

$$\times \left[\int_{R_\sigma} \int_{R_t} \frac{|f(x)g(y)|}{[r_t(x,y)]^{\sigma/q+t/p'}}\, dx^{(t)} dy^{(\sigma)} \right] dx^{(n-t)} dy^{(s-\sigma)}.$$

After this, the proof of inequality (2) reduces to the successive application of inequalities (1′) and (1).

2. Let Ω_n be a bounded domain in R_n, star-shaped with respect to some ball. In the totality C^l of all functions that are l times continuously differentiable in Ω_n, introduce a norm by the formula

$$\left\|r_{n-t}^{-k}(x)u(x)\right\|_{L_p(\Omega_n)} + \left\|r_{n-t}^{-k}(x)D^l u(x)\right\|_{L_p(\Omega_n)}, \tag{4}$$

where $p > 1$; $-\frac{n-t}{p'} < k < \frac{n-t}{p}$.

$$D^l u(x) = \left\{ \sum_{\alpha_1+\alpha_2+\cdots+\alpha_n=l} \left(\frac{\partial^l u(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} \right)^2 \right\}^{1/2}.$$

The closure of the set C^l in the norm (4) will be called the space $W_{p,k}^l(\Omega_n, R_t)$. It is easy to show that the functions obtained under such a completion have generalized derivatives of order l in Ω_n. The norm in the space $W_{p,k}^l(\Omega_n, R_t)$ is defined by formula (4).

Denote $\Omega_s = \Omega_n \cap R_s$. Analogously one may introduce the spaces $W_{q,h}^m(\Omega_s, R_\sigma)$ $(R_\sigma \subset R_s)$ of functions on Ω_s.

With the aid of inequality (2) and the integral representation of S. L. Sobolev ([6], p. 62) of functions from C^l, the following embedding theorem can be proved.

Theorem 2. Let R_n, R_s, R_t, R_σ satisfy the conditions of Theorem 1 and let $k < \frac{s-\sigma}{p}$. Then the space $W_{p,k}^l(\Omega_n, R_t)$ is embedded in the space $W_{q,h}^m(\Omega_s, R_\sigma)$, and the inequality

$$\|u\|_{W_{q,h}^m(\Omega_s, R_\sigma)} \leq c\|u\|_{W_{p,k}^l(\Omega_n, R_t)} \tag{5}$$

holds.

1. If

$$l - \frac{n}{p} + \frac{sk}{s-\sigma} < m < l + k - \frac{n-\sigma}{p},$$

then

$$k < h < \frac{s-\sigma}{\sigma} \left(\frac{n}{p} - l - k + m \right);$$

if, however,

$$l + k - \frac{n-\sigma}{p} \leq m < l - \frac{n-s}{p},$$

then

$$k < h < l + k - m - \frac{n-s}{p},$$

and the number q in both cases is determined by the equality

$$q = \frac{sp}{n - (l + k - m - h)p}.$$

II. If

$$m \leq l - \frac{n}{p} + \frac{sk}{s - \sigma},$$

then inequality (5) is valid for any q and h satisfying the conditions

$$1 < q < \frac{s-\sigma}{k}, \qquad h < \frac{s-\sigma}{q}.$$

Remark. The first part of Theorem 2 carries over without change to the classes of functions in unbounded domains considered in (1). For $s = n$, inequality (5) was obtained under somewhat more general assumptions on the domain Ω_n in the paper of V. P. Il' in (7). Analogues of inequality (5) are also found in the paper of M. I. Vishik (8).

3. The method of proof of inequality (1) presented above makes it possible to obtain inequality (5) for a bounded domain $\Omega_{n-t} = \Omega_n \cap R_{n-t}$ in the case $\sigma = 0$ ($t > 0$) and in the case $s = t$ ($h = 0$), but with

$$q < \frac{sp}{n - (l + k - m - h)p}.$$

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Note: Figure translations are in progress. See original paper for figures.