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Abstract

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MATHEMATICS

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EDGE-VERTEX FUNCTIONS AND DISTRIBUTIVE PROPERTIES OF GRAPHS

(Presented by Academician S. L. Sobolev on 16 III 1961)

1. Here we study an edge-vertex function $\Phi(L)$ of a special kind,* closely connected with the distributive (chromatic) properties of graphs.

Let L be an arbitrary nonempty graph; ab some edge of L ; L_α the graph obtained from L by deleting the edge ab (without deleting vertices); L_β the graph obtained from L by deleting the edge ab and replacing the vertices a and b by a single vertex adjacent to those of the remaining vertices of L that were adjacent to at least one of the vertices a, b ; L_γ the graph obtained from L by deleting the edge ab and replacing the vertices a and b by a single vertex adjacent to those of the remaining vertices of L that were adjacent to both vertices a, b .

Next, let K be a ring (initially free) generated by the elements α, β, γ and e (the unit); $\Phi(L)$ a function on graphs taking values in K and satisfying the conditions

$$\Phi(L) = \alpha\Phi(L_\alpha) + \beta\Phi(L_\beta) + \gamma\Phi(L_\gamma) + e; \quad (1)$$

$$\Phi(E_n) = 0, \quad n = 0, 1, 2, \dots, \quad (2)$$

where E_n is the empty (edgeless) n -vertex graph.

Finally, let Ω denote the necessary and sufficient condition under which the function Φ is determined uniquely from (1) and (2) and takes equal values on almost isomorphic (i.e., isomorphic up to the presence of isolated vertices) graphs; the ring obtained from K by imposing the relations Ω on the generators α, β, γ, e will still be denoted by K .

The method for deriving Ω explicitly and for studying the function Φ determined by (1) and (2) under the condition Ω is the same as in the cases of vertex functions ⁽²⁾ and edge functions ⁽³⁾. We indicate here only the main stages.

2. The condition Ω can be expressed by the system of relations

$$(\alpha - e)\gamma = 0; \quad (3_1)$$

$$\gamma(\alpha + \beta - e) = 0; \quad (3_2)$$

$$\beta\gamma = 0; \quad (3_3)$$

$$(\alpha\beta - \beta\alpha)\alpha^n(\alpha + \beta) = 0; \quad (3_4)$$

$$(\alpha\beta - \beta\alpha)\beta^n(\alpha + \beta) = 0; \quad (3_5)$$

$$(\alpha - e)\alpha^{n+1}\beta(\alpha + \beta) = 0; \quad (3_6)$$

$$(\alpha - e)\alpha\beta^{n+1}(\alpha + \beta) = 0; \quad (3_7)$$

$$\gamma(\alpha - e)\alpha^n(\alpha + \beta) = 0; \quad (3_8)$$

$$\gamma(\alpha - e)\beta^n(\alpha + \beta) = 0; \quad (3_9)$$

$$\gamma\alpha^n\beta(\alpha + \beta) = 0, \quad (3_{10})$$

$$\gamma\beta^{n+1}(\alpha + \beta) = 0, \quad (3_{11})$$

where $n = 0, 1, 2, \dots$

* A general definition of an edge-vertex function is given in ⁽¹⁾, § 9; here we study the case $\delta = \lambda = 0$.

To derive (3₁), (3₂), (3₃), (3₅), (3₇), and (3₁₁) under the assumption of uniqueness of Φ , we compare the results of applying (1) twice to the corresponding graphs in Fig. 1, where the first result is obtained if (1) is applied first to the edge ac , then (where possible) to the edge bc (or bd), and the second if one begins with bc (or bd); in the process of deriving each of these six relations the preceding ones are used. The remaining five relations (3) are obtained as purely algebraic consequences*.

To prove the sufficiency of conditions (3), we first show that with their aid the general expression $\Phi(L)$ is reduced to the form

$$\Phi(L) = [f_L(\alpha) + g_L(\beta) + \alpha h_L(\beta)](\alpha + \beta) + q_L(\gamma) + e, \quad (4)$$

Fig. 1

Figure 1: Fig. 1

where f_L, g_L, h_L, q_L are polynomials with nonnegative integer coefficients, and $g_L(0) = h_L(0) = q_L(0) = 0$; we then prove the uniqueness of Φ , considering three cases of the mutual position of the pair of edges ac and bc (or bd) in an arbitrary graph (Fig. 2), applying induction on the number of edges of the graph and using the relations (3) once more.

Fig. 1

ac and bc (or bd) in an arbitrary graph (Fig. 2), applying induction on the number of edges of the graph and using the relations (3) once more.

3. To clarify the meaning of the coefficients in $f_L(\psi)$, put $\beta = \gamma = 0$; then all relations (3) will be satisfied, and the generating element α will remain free. Equality (4) takes the form $\Phi(L) = \alpha f_L(\alpha) + e$, while equation (1) takes the form $\Phi(L) = \alpha \Phi(L_\alpha) + e$, whence one obtains:

$$f_L(\alpha) = \alpha^{d_2(L)-2} + \alpha^{d_2(L)-3} + \dots + \alpha + e,$$

where $d_2(L)$ is the number of edges of the graph L .

4. If $\alpha = \gamma = 0$, then (3) are satisfied, while β remains free; analogously to the preceding case we obtain:

$$g_L(\beta) = \beta^{B(L)-2} + \beta^{B(L)-3} + \dots + \beta,$$

where $B(L) = d_2(L) - l(L) = d_1(L) - \chi(L)$ is the number of successive contractions of edges (operations of type $L \rightarrow L_\beta$) transforming L into the empty graph; $l(L)$ is the cyclomatic number; $d_1(L)$ is the number of vertices; $\chi(L)$ is the number of connected components of the graph L .

5. For $\alpha = e$, $\gamma = 0$, conditions (3) are satisfied, and β remains free. Equation (1) assumes the form

$$\Phi(L) = \Phi(L_\alpha) + \beta \Phi(L_\beta) + e, \tag{1'}$$

whence

$$\Phi(L) = \sum_{k \geq 0} b_k(L) \beta^k,$$

where

$$b_k(L) = b_k(L_\alpha) + b_{k-1}(L_\beta), \quad k = 1, 2, \dots; \tag{5}$$

$$b_0(L) = b_0(L_\alpha) + 1. \quad (6)$$

To decipher $b_k(L)$, denote by $p_{ji}(L)$ the number of such edge subgraphs M of the graph L (i.e., graphs obtained from L by deleting some edges followed by deleting isolated vertices), for which—

* We have sacrificed the independence of the relations (3) for the sake of technical convenience of transformations (which are omitted here).

for which $d_2(M) = j$, $l(M) = i$. Further, let τ be the number of triangles of the graph L containing the edge ab , and let $p_{ji}^{t+}(L_\beta)$ be the number of those edge subgraphs (with number of edges j and cyclomatic number i) of the graph L_β that contain t edges from among the τ edges obtained as a result of contracting the triangles under the operation $L \rightarrow L_\beta$. Then

$$p_{ji}(L) = p_{ji}(L_\alpha) + \sum_{r+s \leq \tau} C_{r+s}^r \cdot 2^s \cdot p_{j-r-1, i-r}^{r+s}(L_\beta),$$

whence

$$\begin{aligned} & \sum_{i \geq 0} (-1)^i p_{i+k+1, i}(L) = \\ & = \sum_{i \geq 0} (-1)^i p_{i+k+1, i}(L_\alpha) + \sum_{i \geq 0} (-1)^i \sum_{r+s \leq \tau} C_{r+s}^r \cdot 2^s \cdot p_{i+k-r, i-r}^{r+s}(L_\beta); \end{aligned}$$

but

$$\begin{aligned} & \sum_{i \geq 0} (-1)^i \sum_{r+s \leq \tau} C_{r+s}^r \cdot 2^s \cdot p_{i+k-r, i-r}^{r+s}(L_\beta) = \\ & = \sum_{t=0}^{\tau} \sum_{s=0}^t C_t^{t-s} \cdot 2^s \sum_{i \geq 0} (-1)^i p_{i+k-t+s, i-t+s}^t(L_\beta) = \\ & = \sum_{t=0}^{\tau} \sum_{s=0}^t C_t^s \cdot 2^s \sum_{j \geq 0} (-1)^{j-s+t} p_{j+k, j}^t(L_\beta) = \\ & = \sum_{t=0}^{\tau} \left[\sum_{s=0}^t (-1)^{s-t} C_t^s \cdot 2^s \right] \sum_{j \geq 0} (-1)^j p_{j+k, j}^t(L_\beta) = \\ & = \sum_{t=0}^{\tau} \sum_{j \geq 0} (-1)^j p_{j+k, j}^t(L_\beta) = \sum_{j \geq 0} (-1)^j \sum_{t=0}^{\tau} p_{j+k, j}^t(L_\beta) = \sum_{j \geq 0} (-1)^j p_{j+k, j}(L_\beta)^*. \end{aligned}$$

Consequently, if we denote

$$\tilde{b}_k(L) = \sum_{i \geq 0} (-1)^i p_{i+k+1, i}(L),$$

then we shall have

$$\tilde{b}_k(L) = \tilde{b}_k(L_\alpha) + \tilde{b}_{k-1}(L_\beta), \quad k = 1, 2, \dots;$$

moreover,

$$\tilde{b}_0(L) = p_{10}(L) = d_2(L),$$

whence

$$b_0(L) = b_0(L_\alpha) + 1.$$

And since relations (5) and (6) determine the quantities $b_k(L)$ uniquely for a given L , it follows that $b_k(L) = \tilde{b}_k(L)$, i.e.

$$b_k(L) = \sum_{i \geq 0} (-1)^i p_{i+k+1, i}(L), \quad k = 0, 1, 2, \dots \quad (7)$$

From (7) and from equality (4), for $\alpha = e$, $\gamma = 0$ we obtain

$$h_L(\beta) = \frac{\sum_{k \geq 0} \sum_{i \geq 0} (-1)^i p_{i+k+1, i}(L) \beta^k - e}{e + \beta} - g_L(\beta) - [d_2(L) - 1]e.$$

6. For $\alpha = e$, $\beta = 0$, conditions (3) are satisfied, while γ remains free. Equation (1) takes the form

$$\Phi(L) = \Phi(L_\alpha) + \gamma \Phi(L_\gamma) + e, \quad (1')$$

whence, taking (4) into account,

$$q_L(\gamma) = \sum_{k \geq 1} q_k(L) \gamma^k, \quad (8)$$

where $q_k(L)$ is the number of those edge subgraphs M of the graph L for which all connected components are complete graphs and for which $B(M) =$

Fig. 2

Figure 2: Fig. 2

* In replacing the summation index i by j we used the fact that $p_{j+k,j} = 0$ for $j < 0$.

$= k + 1$. This is verified by substituting (8) into (1') and counting the subgraphs M of the indicated type in L , L_α , and L_γ .

7. Let $r_k(L)$ be the number of k -partitions of the graph L ($k = 1, 2, \dots$). As is known ((1), Ch. 2, §§ 3-4 and (4), § 9), the partition polynomial

$$R(L) = \sum_{k \geq 1} r_k(L) z^k \quad (9)$$

satisfies the conditions

$$R(L) = R(L_\alpha) - R(L_\beta), \quad (10)$$

$$R(E_n) = \sum_{k \geq 0} \frac{\Delta^k 0^n}{k!} z^k, \quad n = 0, 1, 2, \dots, \quad (11)$$

$\Delta^k 0^n / k!$ being the coefficient of $z^{(k)} = z(z-1)\dots(z-k+1)$ in the expansion of z^n in generalized powers of z . If we put

Fig. 2

$$\tilde{R}(L) = \sum_{k \geq 1} r_k(L) y^{(k)}, \quad (12)$$

then $\tilde{R}(L)$ will satisfy equation (10) and the initial conditions

$$\tilde{R}(E_n) = y^n \left(= \sum_{k \geq 0} \frac{\Delta^k 0^n}{k!} y^{(k)} \right). \quad (11')$$

Using (5) and (6), one can verify directly that the function

$$\Phi(L) = \sum_{k \geq 0} (-1)^k b_{k-1}(L) y^{d_1(L)-k}$$

(where $b_{-1}(L) = 1$) satisfies equation (10) and the initial conditions (11'); consequently,

$$\tilde{R}(L) = \Phi(L) = \sum_{k \geq 0} (-1)^k b_{k-1}(L) y^{d_1(L)-k},$$

and we obtain an explicit expression for the number of m -partitions of the graph L :

$$r_m(L) = \sum_{k \geq 0} (-1)^k \frac{\Delta^m 0^{d_1(L)-k}}{m!} b_{k-1}(L) = \sum_{j \geq 0} \sum_{k \geq 0} (-1)^{j+k} \frac{\Delta^m 0^{d_1(L)-k}}{m!} p_{j+k,j}(L).$$

The polynomial

$$\tilde{R}(L) = \sum_{k \geq 0} (-1)^k b_{k-1}(L) y^{d_1(L)-k} = \sum_{j \geq 0} \sum_{k \geq 0} (-1)^{j+k} p_{j+k,j}(L) y^{d_1(L)-k}.$$

can be transformed into the form

$$\tilde{R}(L) = \sum_M (-1)^{d_2(M)} y^{\chi(M)}, \quad (13)$$

where the summation is over all graphs M obtained from L by deleting some edges (without deleting vertices). For $y = m$, the quantity $\tilde{R}(L)$, as follows from (12), expresses the number $P(L, m)$ of such m -colorings of L in which not necessarily all m colors are used and colorings differing by a permutation of the colors are counted as distinct; hence, from (13) as well, there follows the formula

$$P(L, m) = \sum_M (-1)^{d_2(M)} m^{\chi(M)},$$

derived in (5) by another method (see also (6)).

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References

1. A. A. Zykov, *Matem. sborn.*, **24**, No. 2 (1949).
2. A. A. Zykov, *Izv. Sibirsk. otd. AN SSSR*, No. 5 (1959).
3. A. A. Zykov, *Izv. Sibirsk. otd. AN SSSR*, No. 9 (1960).

4. A. A. Zykov, *Izv. Sibirsk. otd. AN SSSR*, No. 12 (1960).

5. H. Whitney, *Ann. Math.*, **33**, 688 (1932).

6. W. T. Tutte, *Canad. J. Math.*, **6**, No. 1 (1954).

Note: Figure translations are in progress. See original paper for figures.

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