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Abstract

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ON A CERTAIN CRITERION FOR THE SPECIALNESS OF πd-GROUPS

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§ 1. It is known that those finite groups in which all proper subgroups are invariant have been well studied. It is therefore natural to carry out further investigation of groups by classifying them according to the values of the number of classes of noninvariant subgroups contained in the group.

Groups with a prescribed number of classes of noninvariant subgroups were studied by O. Yu. Schmidt ([1], [2]), D. Sigli ([3], [4]), P. I. Trofimov ([5], [6]), I. Sep and N. Itô ([7], [8]). The study of the properties of groups with a prescribed number of classes of noninvariant subgroups whose orders are divisible, at least, by one prime number from a given fixed nonempty set π of prime divisors of the order of the group (the so-called πd-subgroups ([9])) was undertaken by E. N. Toropov ([10]).

In the present paper we continue the investigation of the influence of the number of classes of noninvariant πd-subgroups of a group G, for a given number of distinct prime π-divisors of its order, on its properties, and present further results in this direction. Special cases of these are Theorems 3–9 of ([6]), Theorems 1–4 of ([10]), and the results of O. Yu. Schmidt concerning the solvability of groups with one and two classes of noninvariant subgroups ([1], [2]).

§ 2. We give the notation, concepts, and definitions used in this paper: π is some nonempty set of prime numbers; G is a finite group of order $g = mn$, where $m > 1$ is the greatest π-Sylow divisor ([11]) of the order g, and $n \geqslant 1$, with

$$n = q_1^{\beta_1} q_2^{\beta_2} \dots q_v^{\beta_v}$$

the canonical decomposition in the case $n > 1$; a πd-group (πd-subgroup) is a group (subgroup) whose order is divisible by some $p \in \pi$ ([9]); t is the number of distinct prime π-divisors of the order, and r is the number of classes of noninvariant πd-subgroups of the group G.

Definition. Put $r - t = \lambda$. Then the numbers r and $\lambda + 2$ will be called, respectively, the π-rank and the π-type of the group.

Thus, πd-groups may be classified according to the values of their π-rank and π-type.

For nonspecial groups, by the Trofimov–Toropov lemma ([6], [10]), the number $\lambda \geqslant -1$. Therefore, if the π-type of a group is not greater than zero, then it is special. Groups having positive values of the π-type may be either special or nonspecial.

Thus, the principal results obtained by us take the following form:

Theorem 1. Every πd-group G of π-type 1 is solvable ([12]).

Theorem 2. Every πd-group G of π-type 2 is solvable.

Theorem 3. Every πd-group G for which $m \neq p$ (p is a prime number) is π-solvable.

In proving these theorems we used the theorems of O. Yu. Schmidt ([2]), Burnside ([13]), and the "intersection method" of P. I. Trofimov (([5]), Lemmas 1 and 1′), which is a further development of the so-called "method of composition of subgroups."

Relying on Theorem 1 and Hall' s theorem ([14]), we obtain the following theorem:

Theorem 4. If G is a nonspecial πd-group of π-rank greater than 1, then its π-type is not less than 2.

Using Theorems 2, 4 and the theorems of P. Hall and S. A. Chunikhin ([15]), we obtain the theorem:

Theorem 5. If G is a nonspecial πd-group of π-rank greater than 3, then its π-type is not less than 3.

Lemma. If G is a nonspecial πd-group of π-rank greater than 3 and of π-type equal to 3, then it contains only one class of noninvariant Sylow πd-subgroups.

On the basis of this lemma, Theorems 3, 5, and S. A. Chunikhin' s theorem, we finally arrive at the following theorem:

Theorem 6. If G is a nonspecial πd-group of π-rank greater than 4, then its π-type is not less than 4.

From this theorem the following very important consequence is obtained:

Corollary. πd-groups of π-types 1, 2, and 3 with π-rank greater than 4 are special.

The significance of this result is seen from the fact that, for such groups, in our preceding Theorems 1, 2, and 3 only their solvability or generalized solvability was shown.

The general conjecture arises that, for every n, one can indicate an r_0 such that all groups of π-type n and of π-rank exceeding r_0 will be special.

In conclusion to the present article I consider it my pleasant duty to express my deep gratitude to my scientific adviser, Corresponding Member of the Academy of Sciences of the BSSR S. A. Chunikhin.

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Note: Figure translations are in progress. See original paper for figures.