ON THE QUESTION OF ESTIMATING THE ERROR IN SOLVING THE DIRICHLET PROBLEM FOR THE LAPLACE EQUATION BY THE METHOD OF GRIDS

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Abstract

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ON THE QUESTION OF ESTIMATING THE ERROR IN SOLVING THE DIRICHLET PROBLEM FOR THE LAPLACE EQUATION BY THE METHOD OF GRIDS

(Presented by Academician S. L. Sobolev on 17 XII 1960)

Error estimates in solving, by the method of grids, the interior Dirichlet problem for the Laplace and Poisson equations, first obtained by S. A. Gershgorin ([1,2]), and then by other authors (see, for example, ([3–8])), contain quantities depending on the unknown exact solution, which substantially hinders their practical application. S. L. Sobolev drew our attention to the fact that sufficiently accurate estimates can be obtained in terms of quantities depending on the approximate solution.

1°. Let in some n-dimensional domain D with boundary S the Laplace equation be considered

$$\Delta u = \sum_{\nu=1}^{n} \frac{\partial^2 u}{\partial x_\nu^2} = 0 \tag{1}$$

with boundary condition $u|_S = f$.

The application of the method of grids to the solution of equation (1) consists, as is known, in the following. We construct in the domain D some grid with step h and replace at the nodes of this grid the Laplace operator Δ by some approximate difference operator Δ_h with approximation accuracy, say, $R(u)$, i.e. we set

$$\Delta_h u = \Delta u + R(u). \tag{2}$$

Let u be the exact solution of equation (1), and $\bar{v}_h$ the numerical solution of the difference equation

$$\Delta_h \bar{v}_h = 0, \qquad \bar{v}_h|_S = f. \tag{3}$$

Construct in the domain $D + S$ some function v_h which is an interpolation function for $\bar{v}_h$, i.e. which coincides with $\bar{v}_h$ at the grid nodes of the domain D and satisfies on the boundary S the condition

$$v_h|_S = f.$$

The aim of our note is to estimate the error which we admit by solving, instead of the exact equation (1), the approximate equation (3), in terms of quantities depending on the interpolation function v_h.

Below we shall rely on the following theorems.

Theorem 1. *Let in an n-dimensional domain D with boundary S two functions u, v be defined, taking on S one and the same value f.*

Suppose, moreover, that the function u is continuous in $D + S$ and harmonic inside D, while the function v is continuous in $D+S$ together with its derivatives up to and including second order, and in the domain D

$$|\Delta v| \leqslant E, \qquad E = \text{const}.$$

Then inside D the inequality holds

$$|u - v| \leqslant \gamma E, \qquad \frac{1}{\gamma} = 2\sum_{\nu=1}^{n} \frac{1}{a_\nu^2},$$

where a_ν $(\nu = 1, 2, \dots, n)$ are the semiaxes of the n-dimensional ellipsoid L containing the domain D.

In proving the theorem we use the auxiliary function

$$z(x_1, x_2, \dots, x_n) = \gamma E \left(1 - \sum_{\nu=1}^{n} \frac{(x_\nu - \bar{x}_\nu)^2}{a_\nu^2}\right),$$

where $\bar{x}_\nu$ $(\nu = 1, 2, \dots, n)$ are the coordinates of the center of the ellipsoid L, and the properties of sub- and superharmonic functions [9].

Corollary of Theorem 1. Suppose that in Theorem 1, instead of a harmonic function, one considers a function u satisfying in the domain D the Poisson equation

$$\Delta u = \varphi(x_1, x_2, \dots, x_n), \tag{4}$$

where the function φ is continuous in $D+S$. Then inside D the inequality holds

$$|u - v| \leqslant \gamma E_1,$$

where

$$E_1 = \max_{D+S} |\varphi - \Delta v|.$$

Theorem 2. Suppose that in the domain D two functions u, v are defined, taking on the boundary S of this domain the values f_1, f_2, respectively.

Suppose, moreover, that the function u is continuous in $D + S$ and harmonic inside D, while the function v is continuous in $D+S$ together with its derivatives up to and including second order, and

$$|\Delta v| \leqslant E \quad \text{in } D, \qquad E = \text{const}.$$

Then in the domain D the inequality

$$|u - v| \leqslant \gamma E + \varepsilon^*, \qquad \frac{1}{\gamma} = 2 \sum_{\nu=1}^{n} \frac{1}{a_\nu^2},$$

is valid, where

$$\varepsilon^* = \max_S |f_1 - f_2|;$$

a_ν $(\nu = 1, 2, \dots, n)$ are the semiaxes of an n-dimensional ellipsoid containing the domain D.

In proving this theorem we rely on Theorem 1 and use the maximum theorem for a harmonic function ([10]).

Corollary of Theorem 2. Suppose that in Theorem 2, instead of a harmonic function, one considers a function u satisfying in the domain D the Poisson equation (4).

Then in D the inequality holds

$$|u - v| \leqslant \gamma E_1 + \varepsilon^*,$$

where

$$E_1 = \max_{D+S} |\varphi - \Delta v|, \qquad \varepsilon^* = \max_S |f_1 - f_2|.$$

Let the interpolating function v_h, constructed in some way, have continuous derivatives up to second order inclusive in the domain $D + S$ and satisfy in D the condition

$$|\Delta v_h| \leq E_h,$$

where E_h is some constant.

Then, using Theorem 1, we obtain for the error of the approximate solution of equation (1) the estimate

$$|u - v_h| \leq \gamma E_h. \tag{5}$$

If, instead of Laplace's equation (1), the Poisson equation is solved in the domain D,

$$\Delta u = \varphi(x_1, x_2, \dots, x_n), \qquad u|_S = f,$$

then the error estimate for the approximate solution, by virtue of the corollary to Theorem 1, is expressed by the formula

$$|u - v_h| \leq \gamma \max_{D+S} |\varphi - \Delta v_h|.$$

We note that in the case where the function v_h does not coincide with u on the boundary of the domain, to estimate the error of the approximate solution one should use Theorem 2 or the corollary of this theorem, depending on whether Laplace' s equation or Poisson' s equation is being solved.

If the values of the function v_h at the interior nodes of the domain D differ somewhat from $\bar{v}_h$, then to estimate inside D the difference $|u - \bar{v}_h|$ it is necessary to use the inequality

$$|u - \bar{v}_h| \leq \max_{D} |u - v_h| + \max_{D} |v_h - \bar{v}_h|. \tag{6}$$

Remark 1. The quantity E_h in formula (5) can also be determined by using Theorem 1 of V. S. Ryaben' kii ([1], p. 158), which gives estimates for the derivatives of the interpolating function v_h in terms of differences of the corresponding orders of the function prescribed at the grid nodes.

Remark 2. The theorems formulated above make it possible to estimate the error of an approximate solution of Laplace' s equation or Poisson' s equation obtained by other approximate methods, different from difference methods.

2°. As an example, consider the two-dimensional Dirichlet problem in the domain D, which is the ellipse $x^2+4y^2 = 1$, with the following boundary conditions

$$u = \begin{cases} \exp\left[\frac{\pi}{4}\sqrt{1-x^2}\right]\cos\frac{\pi}{2}x, & \text{for } 0 \le y \le 0.5, \\ \exp\left[-\frac{\pi}{4}\sqrt{1-x^2}\right]\cos\frac{\pi}{2}x, & \text{for } -0.5 \le y < 0. \end{cases}$$

The exact solution of the problem is the function

$$u = \exp\left[\frac{\pi}{2}y\right]\cos\frac{\pi}{2}x.$$

Solving this problem by the grid method with step $h = 0.5$, using the simplest difference Laplace operator, we find $\bar{v}_h(0,0) = 1.022446$, $\bar{v}_h(0.5,0) = 0.720283$. The corresponding exact values are 1 and 0.707107.

To estimate the error by our method, we construct the interpolating function v_h in the form

$$v_h(x,y) = e^{\alpha y}(a + bx^2 + cx^4),$$

where the parameters a and α of the function $ae^{\alpha y}$ are determined by the method proposed by Aran ([12]), and the coefficients b and c are found by the method of least squares.

As a result of the computations we obtain

$$v_h(x,y) = e^{1.554239y}\left(1.010063 - 1.208807x^2 + 0.198743x^4\right),$$

and at the interior nodes of the grid

$$\|v_h - \bar{v}_h\| \leqslant 0.012383,$$

on the boundary of the domain D

$$\|v_h - u\| \leqslant 0.015841,$$

and inside D

$$\|\Delta v_h\| \leqslant 0.275744.$$

Thus, by virtue of Theorem 2 and inequality (6), we obtain the following estimate of the error of the approximate solution:

$$\|u - \bar{v}_h\| \leqslant 0.055799,$$

whereas the maximum true error is equal to 0.022446.

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CITED LITERATURE

[1] S. A. Gershgorin, *Izv. Leningrad. Politekhn. Inst.*, **30**, 75 (1927).
[2] S. A. Gershgorin, *Zhurn. Prikl. Fiz.*, **6**, issue 3-4, 3 (1929).
[3] A. A. Abramov, *Vychisl. Matem. i Vychisl. Tekhn.*, **1**, 37 (1953).
[4] E. A. Volkov, *DAN*, **96**, No. 5 (1954).
[5] L. A. Lyusternik, *UMN*, **9**, issue 2 (1954).
[6] Sh. E. Mikeladze, *Numerical Methods for Integrating Differential Equations with Partial Derivatives*, USSR Academy of Sciences Press, 1936.
[7] Sh. E. Mikeladze, *Izv. AN SSSR, Ser. Matem.*, **5**, No. 1, 57 (1941).
[8] M. R. Shura-Bura, *Vychisl. Matem. i Vychisl. Tekhn.*, **1** (1953).
[9] I. G. Petrovsky, *UMN*, issue 8, 107 (1941).
[10] S. L. Sobolev, *Equations of Mathematical Physics*, Moscow, 1954.
[11] V. S. Ryabenkii, A. F. Filippov, *On the Stability of Difference Equations*, 1956.
[12] S. Arend, *Ciel et terre*, **71**, No. 5-6 (1955).

Note: Figure translations are in progress. See original paper for figures.