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Abstract

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FORMAL DEFINITION OF THE R-OPERATION

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The R-operation was first introduced by N. N. Bogolyubov and O. S. Parasyuk in the work ([1]) as a generalization of the results of Dyson–Salam ([2,3]). The fundamental analytic theorem on the absence of divergences in any order of perturbation theory after application of the R-operation was proved in ([4]) and then considered in detail in ([5–7]); a further refinement of the proof is contained in ([8]). This theorem is the basis for a variety of applications of the theory, in particular for a principled solution of the question of the possibility of renormalizing one or another particular theory of interaction ([9,10]).

In the present note an attempt is made to give a simple, formal definition of the R-operation, convenient for concrete applications, and then to indicate explicitly the connection between the structure of the R-operation and arbitrary quasilocal operators allowed by the conditions of causality and unitarity. For simplicity of exposition it is convenient to make use, without any restriction of generality, of the case of spinor electrodynamics. Moreover, for definiteness, let us suppose that the regularization of causal functions being used has the property of Lorentz covariance. One may imagine, for example, the causal functions regularized by the Pauli–Villars method.*

Let us first introduce some definitions. We shall call an elementary graph, and denote by $\sigma(x_i)$, or simply by σ_i, a vertex of first order together with the lines belonging to it. In the case under consideration there are three such lines: photon, electron, and positron. The fact that all lines belonging to an elementary graph are pairwise distinct has a certain advantage in the sense of simplifying the exposition. This restriction, however, is very easy to remove, as will be clear from what follows.

We shall say that a graph G belongs to the class $\mathfrak{G}$ if and only if it possesses the following two properties:

1°. All vertices of the graph G are elementary graphs.

2°. All internal lines of the graph G are obtained by pairwise coupling lines belonging to distinct, and only distinct, vertices, with observance of the principle of continuity of fermion cycles.

For applications in quantum field theory, it is precisely classes of type $\mathfrak{G}$ that are essential; therefore we shall restrict ourselves to consideration of graphs only of this type.

Let us now define the operation of multiplication of two graphs $G' \in \mathfrak{G}$ and $G'' \in \mathfrak{G}$ in such a way that, as a result of carrying it out, one again obtains a graph G belonging to the same class $\mathfrak{G}$. To this end, establish between the external lines of the graphs G' and G'' some arbitrary correspondence satisfying only requirement 2°. As a result of pairwise coupling of the corresponding lines, a new graph G arises, which we shall call the product of the graphs G' and G''. Obviously, the order of the graph G is equal to the sum of the orders of the factor graphs G' and G''. It is easy to verify that, as the graphs G' and G'' and the law of correspondence compatible with 2° are specified, one-

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* The assumption of Lorentz invariance of the regularization is not necessary (see ([11,12])).

uniquely determines the graph $G = G' * G''$, and, conversely, specifying the graph G uniquely determines the correspondence between the external lines of the graphs G' and G'' into which the graph G can be split by cutting the appropriate internal lines. This assertion is valid, of course, only in the case when all lines of the elementary graph are pairwise distinct.

In the case of three graphs G', G'', G''', their product is defined successively: first the product of the first two graphs $G' * G''$ is formed, and then the resulting graph is multiplied by the third, $(G' * G'') * G'''$.

It is clear that the product defined in this way has the property of commutativity

$$G' * G'' = G'' * G'$$

and associativity

$$(G' * G'') * G''' = G' * (G'' * G''').$$

The latter relation is, in essence, trivial, since the graph $G' * (G'' * G''')$ also determines a correspondence between the external lines of any pair of the three given factor graphs, while the product $(G' * G'') * G'''$ is formed according to the rule determined by this correspondence.

Let us now consider an arbitrary graph of order n, $G_n \in \mathfrak{G}$. In view of the definitions made, one can write the equality

$$G_n = \prod_{(1 \le i \le n)} \sigma_i.$$

The product appearing on the right-hand side of this equality is formed according to the rule determined by the graph G_n. Let us split the set of indices $1, 2, \dots, n$ into m subsets $(i_1, \dots, i_{n_1})$, $(i_{n_1+1}, \dots, i_{n_1+n_2})$, ..., $(i_{n_1+\dots+n_{m-1}+1}, \dots, i_n)$, with $n_1, n_2, \dots, n_m$ indices in each

$$\left(\sum_{1 \le k \le m} n_k = n \right).$$

Denote such a partition by $\mathfrak{N}_m$. Then, by virtue of the indicated properties of the product, we shall also have

$$G_n = (\sigma_{i_1} * \dots * \sigma_{i_{n_1}}) * (\sigma_{i_{n_1+1}} * \dots * \sigma_{i_{n_1+n_2}}) * (\sigma_{i_{n_1+\dots+n_{m-1}+1}} * \dots * \sigma_{i_n}) \equiv \prod_{(1 \le k \le m)} G_{n_k}, \tag{1}$$

whatever the partition $\mathfrak{N}_m$ may be. Let us emphasize that the graphs

$$\sigma_{i_1} * \dots * \sigma_{i_{n_1}} = G_{n_1}, \dots, \sigma_{i_{n_1+\dots+n_{m-1}+1}} * \dots * \sigma_{i_n} = G_{n_m}$$

are uniquely determined. In accordance with the terminology of papers ([1], [4–10]), these partial graphs may be called **generalized vertices**.

For the formulation of the generalized Feynman–Dyson correspondence rules it is convenient to use the symbols $\Delta(G_{n_k})$ introduced in ([1]). By definition, the symbol $\Delta(G_{n_k})$ means that to the graph G_{n_k} there is assigned a certain quasilocal function $d_{G_{n_k}}(x_{i_{n_1+\dots+n_{k-1}+1}}, \dots, x_{i_{n_1+\dots+n_k}})$ with arguments (indices) conforming to the external lines of the graph G_{n_k}. These arguments may be free (if the corresponding external lines of G_{n_k} are also external lines of G_n) or occupied (if the corresponding external lines of G_{n_k} are internal lines of G_n) by the corresponding regularized causal functions $\operatorname{reg} D^c(x_j - x_l)$ or $\operatorname{reg} S^c(x_j - x_l)$.

We shall now define the quasilocal function $d_{G_{n_k}}$ as follows. If the graph G_{n_k} $(n_k \ge 2)$ is disconnected or weakly connected, then, by definition,

we shall set

$$d_{G_{n_k}} \equiv 0.$$

If the graph G_{n_k} is connected, then the quasilocal function $d_{G_{n_k}}$ should be required to have the proper transformation properties (in space-time, in isotopic space, etc.). Its exact form will be determined only after passing to the physical limit, as will be discussed below.

Finally, for $n_k = 1$, d_σ is the Dirac vertex matrix.

Let us now associate with the partition (1) the coefficient function

$$\Delta(G_{n_1}) \dots \Delta(G_{n_m}). \tag{2}$$

Multiplying it, in the sense of the normal product, by the field operators corresponding to the external lines of G_n, we obtain an operator corresponding to the graph G_n and to the partition $\mathfrak{N}_m$. Thus, to any graph G_n there corresponds a series of operators of one and the same operator structure (determined by the external lines of G_n), differing only, depending on the form of the partition $\mathfrak{N}_m$ $(1 \le m \le n)$, in their coefficient functions.

As is known ([9,10]), the most general expression for the n-th term in the perturbation-theory expansion of the S-matrix has the form:

$$S_n(x_1, \dots, x_n) =$$

$$= \sum_{(\mathfrak{N}m,\, 1 \le m \le n)} i^m T(\Lambda_{n_1}(x_{i_1}, \dots, x_{i_{n_1}}) \dots \Lambda_{n_m}(x_{i_{n_1+\dots+n_{m-1}+1}}, \dots, x_{i_n})), \tag{3}$$

where the summation extends over all possible distinct partitions $\mathfrak{N}_m$ of the set of variables $x_1, \dots, x_n$. Expression (3) satisfies the basic physical requirements for arbitrary quasilocal Hermitian operators Λ_{n_k}. The R-operation consists, in essence, in a certain particular choice of them, which will now be indicated. Consider the set of all graphs of order n_k: $\{\sigma_1 * \dots * \sigma_{n_k}\}$. By the rule given above, form the set of all coefficient functions $\{d_{\sigma_1 * \dots * \sigma_{n_k}}\}$ corresponding to the graphs in the set $\{\sigma_1 * \dots * \sigma_{n_k}\}$, and then, with the aid of these coefficient functions, the set of all corresponding operators. We now define the operator Λ_{n_k} as the sum of all the operators constructed in this way. In the case when among the lines of an elementary graph there are identical ones, an expression arises leading to the fact that different operators enter the sum with certain integer coefficients equal to the order of the expression.

It is easy to see that a generalized Wick theorem holds. Namely, the T-product

$$T(\Lambda_{n_1}(x_{i_1}, \dots, x_{i_{n_1}}) \dots \Lambda_{n_m}(x_{i_{n_1+\dots+n_{m-1}+1}}, \dots, x_{i_n}))$$

is equal to the sum over all possible graphs of n-th order $G_n \in \mathfrak{G}$, for a given partition $\mathfrak{N}_m$, of all operators with coefficient functions from the set

$$\{\Delta(\sigma_{i_1} * \dots * \sigma_{i_{n_1}}) \dots \Delta(\sigma_{i_{n_1+\dots+n_{m-1}+1}} * \dots * \sigma_{i_n})\}.$$

Thus, expression (3) is the sum over all possible $G_n \in \mathfrak{G}$ and over all distinct partitions $\mathfrak{N}_m$ $(1 \le m \le n)$ of all operators obtained from the coefficient functions (2), and, consequently, the coefficient functions $S_n(x_1, \dots, x_n)$ with the quasilocal functions defined above with the operators Λ_{n_k} have the form

$$\sum_{(\mathfrak{M}m,\,1\leq m\leq n)} \Delta(G_{n_1})\ldots\Delta(G_{n_m}).$$

In (4^{-8}) it is proved that the quasilocal functions $d_{G_{n_k}}$ can be chosen so that the last expression becomes finite after removal of the regularization. Therefore what has been said above may be briefly summarized as follows. The coefficient function

$$R_{G_n}(x_1,\ldots,x_n) = \sum_{(\mathfrak{M}m,\,1\leq m\leq n)} \Delta(G_{n_1})\ldots\Delta(G_{n_m}) \tag{4}$$

with the $\Delta(G_{n_k})$ chosen in the manner just indicated will, after removal of the regularization, contain no divergences; moreover, all the physical conditions imposed on the S-matrix turn out to be automatically satisfied. Expression (4) is the formal definition of the R-operation as an operation assigning to each graph G_n $(n \geq 1)$ its own coefficient function.

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Note: Figure translations are in progress. See original paper for figures.