PHYSICS

G. I. ZELTSER

1961

SovietRxiv

Abstract

Full Text

PHYSICS

G. I. ZELTSER

ON THE DETERMINATION OF THE TOTAL ANGULAR MOMENTUM OF A SYSTEM OF n PARTICLES FOR THE CONFIGURATION j^n

(Presented by Academician Ya. B. Zel' dovich, December 21, 1960)

1. In the theory of atomic or nuclear shells, the question of the values of the total angular momentum J possible in a system of n identical particles, each of which has the same individual angular momentum j, and of the number c_J of distinct states with a given J, is of great interest. The totality of these data is called the J-structure of the shell.

In establishing the J-structure, the properties of the permutation symmetry of the wave function of the system must be taken into account. The most usual requirement is that of antisymmetry of the state, but one also has to deal with more complicated types of symmetry (for example, for the coordinate function in atomic LS-coupling or for the coordinate-spin function in the theory of isotopic spin).

One convenient method for determining the J-structure is the "chain" method, proposed by Jahn ([1]) and systematically applied by Flowers ([2]). In this method it proves necessary to calculate jointly the J-structures belonging to all possible types of permutation symmetry (Young diagrams). For the applicability of the method it is essential that certain different Young diagrams (for one and the same j) be equivalent in their J-structure. The following equivalence relations are used: 1) coincidence of the J-structures for Young diagrams differing by the addition of one or several columns of length $k = 2j + 1$; 2) coincidence of the J-structures for two Young diagrams that complement one another to a rectangular diagram of height k.

However, when only these equivalences are used, calculations for large values of j prove rather difficult. Meanwhile, there exist further equivalences of Young diagrams with respect to the J-structure, relating certain mutually associated diagrams for different values of j. They appreciably simplify the calculations, making it possible, in particular, to reduce the computation of the antisymmetric J-structure for a certain value of j to the computation of a symmetric J-structure for a smaller value of j.

In the present work these equivalences, which, as it seems to us, are also of independent interest, will be obtained on the basis of explicit formulas defining the J-structures through the characters of the corresponding representations of the three-dimensional rotation group R_3.

2. The wave function $\psi(jm)$ of one particle with angular momentum j may be regarded as a vector in a $k=(2j+1)$-dimensional space. Under three-dimensional rotations it is transformed according to the irreducible representation D_j of the group R_3. The wave function of n particles, each of which has angular momentum j, may be regarded as a tensor of rank n in a k-dimensional space. We shall assume that this tensor, $\Psi_{\{\lambda\}}$, has a definite permutation symmetry $\{\lambda\} \equiv \{\lambda_1, \lambda_2, \dots, \lambda_k\}$, i.e., is transformed under permutations of the particles according to the irreducible representation of the symmetric group S_n, corresponding

corresponding partition

$$n = \lambda_1 + \lambda_2 + \dots + \lambda_k \quad (\lambda_1 \geqslant \lambda_2 \geqslant \dots \lambda_k \geqslant 0). \tag{1}$$

The representation D_j in the space of the functions $\psi(jm)$ induces in the space of tensors $\Psi_{\{\lambda\}}$ a representation of the rotation group R_3, generally reducible, which we shall denote by $\Gamma_{\{\lambda\}}$. Its decomposition into irreducible representations

$$\Gamma_{\{\lambda\}} = \sum_J c_J^{\{\lambda\}} D_j$$

(a direct sum) also determines the J-structure of the tensor of the given symmetry $\{\lambda\}$. The coefficients $c_J^{\{\lambda\}}$ are determined by the characters $\chi_{\{\lambda\}}(\varphi)$ of the representation $\Gamma_{\{\lambda\}}$ from the formula

$$c_J^{\{\lambda\}} = \int \chi_{\{\lambda\}}(\varphi) \chi_J(\varphi)\, d\tau, \tag{2}$$

where φ is the rotation angle specifying a class of the group R_3; $\chi_J(\varphi) = \dfrac{\sin(J+1/2)\varphi}{\sin\varphi/2}$ is the character of an irreducible representation of this group; $d\tau$ is the invariant element of its volume, normalized to the group volume, equal to 1 $\left(d\tau = \dfrac{2}{\pi}\sin^2\dfrac{\varphi}{2}\,d\varphi,\ 0 \leqslant \varphi \leqslant \pi\right)$. To determine the J-structure it is necessary to know the characters $\chi_{\{\lambda\}}(\varphi)$, and the coincidence of these characters is sufficient for the coincidence of the J-structures.

The character $\chi_{\{\lambda\}}(\varphi)$ is the trace of the matrix transforming the tensor $\Psi_{\{\lambda\}}$, when the basic k-dimensional space is transformed according to the representation D_j. This representation is a subgroup of the k-dimensional unitary group $U(k)$. It is known, however, that for this latter group tensors of a definite permutation symmetry $\{\lambda\}$ are objects of its irreducible representation. Thus, if we subject the one-particle functions to a general unitary transformation of the group $U(k)$, then the tensors $\Psi_{\{\lambda\}}$ will transform according to an irreducible representation of this group, whose character is equal to [3]

$$\chi_{\{\lambda\}}(\varphi_1, \varphi_2, \dots, \varphi_k) = \frac{\left|z^{l_1}, \dots, z^{l_k}\right|}{\left|z^{k-1}, \dots, z^0\right|}, \tag{3}$$

where $\left|z^{l_1}, \dots, z^{l_k}\right|$ is a determinant of order k, in the p-th row of which stand $z_p^{l_1}, \dots, z_p^{l_k}$,

$$z_p = e^{i\varphi_p} \quad (p = 1, \dots, k) \tag{4}$$

are the characteristic numbers of the matrices of the group $U(k)$;

$$l_p = \lambda_p + k - p \quad (p = 1, \dots, k). \tag{5}$$

The dimension $N_{\{\lambda\}}$ of an irreducible representation of the unitary group is

$$N_{\{\lambda\}} = \frac{D(l_1, \dots, l_k)}{D(k-1, \dots, 0)}, \tag{6}$$

where

$$D(l_1, \dots, l_k) \equiv \begin{vmatrix} l_1^{k-1} \dots l_k^{k-1} \\ \cdot \;\; \cdot \;\; \cdot \;\; \cdot \;\; \cdot \\ 1 \dots 1 \end{vmatrix} = \prod_{p'>p=1}^{k} (l_p - l_{p'}) \tag{7}$$

is the Vandermonde determinant.

Returning from the general k-dimensional unitary group to the irreducible representation D_j of the group R_3, we obtain the required character $\chi_{\{\lambda\}}(\varphi)$ by putting in (3)

$$z_p = e^{i(j+1-p)\varphi} = e^{im\varphi} \quad (p = 1, \dots, k;\; m = j, \dots, -j), \tag{8}$$

i.e., choosing the characteristic numbers corresponding to the irreducible representation D_j.* After substituting (8), the right-hand side of formula (3) is easily reduced to the ratio of two Vandermonde determinants. Taking out from the p-th column of the determinant in the numerator the factor $e^{-il_p p\varphi}$, and from the p-th column of the determinant in the denominator the factor $e^{-ij(k-p)\varphi}$, after a short calculation, taking into account $k = 2j+1$, (1), and (5), we obtain

$$\chi_{\{\lambda\}}(\varphi) = e^{-i\frac{(k-1)}{2}n\varphi} \frac{D\left(e^{il_1\varphi}, \dots, e^{il_k\varphi}\right)}{D\left(e^{i(k-1)\varphi}, \dots, 0\right)}. \tag{9}$$

Expressing the differences of exponentials entering the function D through sines, we find

$$\chi_{\{\lambda\}}(\varphi) = \prod_{p'>p=1}^{k} \frac{\sin(l_p - l_{p'})\varphi/2}{\sin(p'-p)\varphi/2}. \tag{10}$$

For the investigation of this formula it is convenient to write it in a symbolic form analogous to the notation (2). Let

$$D_s(l_1, \dots, l_p) \equiv \prod_{p'>p=1}^{k} \sin(l_p - l_{p'})\frac{\varphi}{2}. \tag{11}$$

Then

$$\chi_{\{\lambda\}}(\varphi) = \frac{D_s(l_1, \dots, l_k)}{D_s(k-1, \dots, 0)}. \tag{12}$$

The structure of this expression is completely similar to the structure of formula (6) for the dimension of an irreducible representation of the group $U(k)$. Formula (12) (or (10)) together with formula (2) completely determines the J-structure.

3. Let us now consider special cases of formula (12) and establish the coincidence of certain J-structures. To give the formulas a more transparent form, it is convenient to introduce the symbolic notations (for integers $m > 0$)

$$m_s \equiv \sin m\frac{\varphi}{2}, \qquad m_s! \equiv \prod_{\mu=1}^{m} \sin\mu\frac{\varphi}{2}, \qquad 0_s! \equiv 1. \tag{13}$$

Then, for example, for the case of one particle, $n = 1$, $\{\lambda\} = \{10\dots0\}$, we obtain from formulas (12) and (5)

$$\chi_{\{1\}}(\varphi) = \frac{D_s(k, k-2, \dots, 0)}{D_s(k-1, k-2, \dots, 0)} = \frac{2_s \cdot 3_s \cdots k_s}{1_s \cdot 2_s \cdots (k-1)_s} = \frac{k_s}{1_s} = \frac{\sin k\varphi/2}{\sin\varphi/2} = \frac{\sin(j+\frac{1}{2})\varphi}{\sin\varphi/2}$$

—an expression already given above for the character of the irreducible representation of the group R_3.

For a **symmetric** tensor of rank n ($\{\lambda\} = \{n\},\ \lambda_1 = n,\ \lambda_2 = \dots = \lambda_k = 0$), we obtain from formula (12), in the notation (13),

$$\chi_{\{n\}}(\varphi) = \frac{(n+k-1)_s!}{n_s!\,(k-1)_s!}. \tag{14}$$

Fig. 1

Figure 1: Fig. 1

For an **antisymmetric** tensor of rank n ($\{\lambda\} \equiv \{1^n\},\ \lambda_1 = \lambda_2 = \cdots = \lambda_n = 1,\ \lambda_{n+1} = \cdots = \lambda_k = 0$), we find

$$\chi_{\{1^n\}}(\varphi) = \frac{k_s!}{n_s!\,(k-n)_s!}. \tag{15}$$

* In another form, an analogous consideration was carried out by E. D. Trifonov (4).

Putting $k = n + k' - 1$ in formula (15) and comparing with (14), we become convinced that the J-structure of an antisymmetric tensor of rank n in a k-dimensional space coincides with the J-structure of a symmetric tensor of the same rank in a k'-dimensional space*. In other words, the J-structure of an antisymmetric configuration of n particles with individual angular momenta j_F coincides with the J-structure of a symmetric configuration of n particles with individual angular momenta j_B, under the condition

Fig. 1

$$j_F = j_B + \frac{n-1}{2}. \tag{16}$$

This equivalence makes it possible to reduce the calculation of the J-structure for antisymmetric configurations to the calculation of the J-structure of symmetric configurations with smaller individual angular momentum. One can also obtain a more general result. Considering the Young diagram shown in Fig. 1 and the corresponding partition

$$\{\lambda'\} = \{p \ldots pr \ldots r0 \ldots 0\}, \tag{17}$$

we obtain for it, by formula (12), in the notation (13),

$$\chi_{\{\lambda'\}}(\varphi) = \prod_{i=1}^{r} \frac{[(r-i)_s!]^2 (p+q-r-i)_s!(p+k-i)_s!}{(p-i)_s!(p+q-i)_s!(q-i)_s!(k-q+r-i)_s!}. \tag{18}$$

This expression is invariant under passage to the associated diagram $\{\tilde{\lambda}'\}$ (which corresponds to the replacement $p \leftrightarrow q$), if at the same time one passes from dimension k to dimension $\tilde{k} = k + p - q$. In other words, for diagrams of this kind the J-structure of the associated diagrams coincides under the condition

$$\tilde{j} = j + \frac{p-q}{2}. \tag{19}$$

Condition (16) is a special case of condition (19), which is obtained if one puts, in diagram (17), $p = n,\ q = r = 1$.

Leningrad Agricultural Institute

Received
17 XII 1960

CITED LITERATURE

1. H. A. Jahn, Proc. Roy. Soc., A201, 516 (1950).

2. B. H. Flowers, Proc. Roy. Soc., A212, 248 (1952).

3. G. Weyl, *Classical Groups, Their Invariants and Representations*, IL, 1949.

4. E. D. Trifonov, Vestn. LGU, No. 22, issue 4, 157 (1958).

* Let us also note that formula (15) is invariant under the replacement $n \leftrightarrow k - n = 2j + 1 - n$ (replacement of particles by "holes"), while formula (14) is invariant under the replacement $n \leftrightarrow k - 1 = 2j$.

Note: Figure translations are in progress. See original paper for figures.