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Abstract

Full Text

MATHEMATICS

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**ON VOLTERRA OPERATORS OBTAINED BY
PERTURBING SELF-ADJOINT OPERATORS**

(Presented by Academician A. N. Kolmogorov on 16 III 1961)

1. We shall adhere to the definitions and notation proposed in ⁽¹⁾.

Recently a number of papers ⁽²⁻⁴⁾ have appeared in which the relationship between the spectral properties of the Hermitian components of a Volterra operator is investigated. In the present note further results in this direction will be indicated, obtained by a method based on theorems on entire functions from the author's note ⁽⁵⁾.

We shall need the following propositions.

A. *If an entire function $f(\lambda)$ admits the lower estimate*

$$|f(\lambda)| \geq \exp \left\{ -T \left(\frac{r}{|\sin \theta|} \right) \right\} \quad (r = |\lambda|, \quad \theta = \arg \lambda),$$

where $T(t)$ is a nondecreasing nonnegative continuously differentiable function, $T(0) = T'(0) = 0$, $f(0) = 1$, then for any p ($1 < p < 2$):

$$\sum_m \frac{1}{|a_m|^p} \leq C_p \int_0^\infty \frac{T(t)}{t^{1+p}} dt,$$

where a_m are the zeros of the function $f(\lambda)$.

By $L(r)$ (or $L_1(r)$) in what follows we shall denote a slowly varying function, i.e., a function for which $L(r) > 0$ and $rL'(r)/L(r) \rightarrow 0$ as $r \rightarrow \infty$.

B. *If an entire function $f(\lambda)$ for some $\rho > 1$ admits the lower estimate*

$$|f(\lambda)| \geq C \exp \left\{ - \left(\frac{r}{|\sin \theta|} \right)^\rho L_1 \left(\frac{r}{|\sin \theta|} \right) \right\},$$

then $n(r) = O(r^\rho L_1(r))$, where $n(r)$ is the number of zeros of the function $f(\lambda)$ lying in the circle $|\lambda| \leq r$.

Assuming that the zeros $\{a_j\}$ of the entire function $f(\lambda)$ are arranged in order of increasing moduli, one may assert that the last relation is equivalent to the following:

$$|a_n|^{-1} = O(n^{-1/\rho}L(n)),$$

where $L(r)$ is a slowly varying function determined from the condition that the functions

$$\varphi_1(r) = r^\rho L_1(r) \quad \text{and} \quad \varphi_2(r) = r^{1/\rho} L(r)$$

are mutually inverse for sufficiently large r .

C. If an entire function $f(\lambda)$ admits the lower estimate

$$|f(\lambda)| \geq C \exp \left\{ - \left(\frac{r}{|\sin \theta|} \right)^\rho \right\} \quad (\rho < 1),$$

then

$$\int_{-\infty}^{\infty} \frac{|\ln |f(t)||}{1+t^2} dt < \infty, \quad \overline{\lim}_{|\lambda| \rightarrow \infty} \frac{\ln |f(\lambda)|}{|\lambda|} < \infty. \quad (1)$$

On the basis of a theorem of M. Cartwright (see ⁽⁶⁾, p. 315), the relations (1) imply the existence and finiteness of the limits

$$\lim \frac{n}{a_n^+} = \lim \frac{n}{a_n^-},$$

where a_n^+ and a_n^- are, respectively, the positive roots and the moduli of the negative roots of the function $f(\lambda)$, numbered in increasing order.

Proposition A is obtained from the result placed at the end of the paper ⁽⁵⁾, if in it one sets $\alpha(t) = p$, $k = 1$.

Proposition B is proved in the same way as the theorem in ⁽⁵⁾.

Proposition C, close to the corollary in ⁽⁵⁾, is derived from the theorem of this paper somewhat more complicatedly.

Theorem 1. Let $A = G + iH$ ($G = \frac{1}{2}(A + A^*)$) be a Volterra operator. If $H \in \mathfrak{S}_p$ ($1 < p < \infty$), then $G \in \mathfrak{S}_p$ and

$$|G|_p \leq C_p |H|_p, \quad (2)$$

where C_p is a certain constant depending only on p . Moreover, if by C_p one understands the least constant in (2), then

$$C_p^* = C_q \quad (p^{-1} + q^{-1} = 1).$$

A weaker assertion was known to I. Ts. Gokhberg and M. G. Krein (see ⁽⁷⁾).

Theorem 2. Let $A = G + iH$ be a Volterra operator. Then, if $s_n(H) = O(n^{-1/\rho}L(n))$, where $1 < \rho < 2$, then $s_n(G) = O(n^{-1/\rho}L(n))$.

We outline the proofs of these theorems. Putting $R_\lambda(G) = (I - \lambda G)^{-1}$ and noting that, under the hypotheses of Theorems 1 and 2, $[HR_\lambda(G)]^2 \in \mathfrak{S}_1$, consider the function

$$f(\lambda) = 1/\det(I + \lambda^2(HR_\lambda(G))^2). \quad (3)$$

Since

$$I + \lambda^2(HR_\lambda(G))^2 = (I - \lambda A^*)(I - \lambda G)^{-1}(I - \lambda A)(I - \lambda G)^{-1},$$

and A is a Volterra operator, the denominator in (3) does not vanish and $f(\lambda)$ is an entire function. It can be shown that $f(\lambda)$ vanishes only at the characteristic numbers of the operator G , and the multiplicity of each root of $f(\lambda)$ is equal to twice the multiplicity of the corresponding characteristic number of G . In addition, using the inequalities of H. Weyl⁽⁸⁾ and the fact that

$$s_j(HR_\lambda) \leq s_j(H)|R_\lambda| \leq s_j(H)/|\sin \theta|,$$

we easily obtain*

$$\frac{1}{|f(\lambda)|} \leq \prod_{j=1}^{\infty} \left(1 + \frac{r^2}{|\sin \theta|^2} s_j^2(H) \right). \quad (4)$$

Under the conditions imposed on $s_j(H)$ in Theorems 1 and 2, the right-hand side of (4) can be estimated so that Propositions A and B will be applicable to the function $f(\lambda)$. Hence Theorem 2 follows, and also Theorem 1 for $1 < p < 2$. For $p = 2$, Theorem 1 coincides with the theorem of L. A. Sakhnovich⁽²⁾ (which can also be obtained analytically, without using Zermelo's axiom, in contrast to^(2, 3)). For $p > 2$, Theorem 1 and the equality $C_p = C_q$ ($p^{-1} + q^{-1} = 1$) can be derived from the case $1 < p < 2$ with the aid of the general Theorem 2 from⁽¹⁾.

We note that, under the hypotheses of Theorem 2, one can prove the inequality

$$\overline{\lim} n^{1/\rho} s_n(G)[L(n)]^{-1} \leq C \overline{\lim} n^{1/\rho} s_n(H)[L(n)]^{-1},$$

where the constant C depends only on ρ and $L(r)$.

2. Let us formulate three theorems on Volterra operators obtained by perturbing a nonnegative operator.

Theorem 3. Let $A = F + T$ be a Volterra operator, where $F = F^*$ is a nonnegative operator, and $T \in \mathfrak{S}_p$ ($p > 1/2$). Then $F \in \mathfrak{S}_p$ and

$$|F|_p \leq C'_p |T|_p,$$

where the constant C'_p depends only on p .

* V. B. Lidskii drew attention to the possibility of applying H. Weyl's inequalities in estimates of this kind.

Theorem 4. Let $A = F + T$ be a Volterra operator, where $F = F^*$ is a nonnegative operator, and $s_n(T) = O(n^{-1/\rho}L(n))$ ($1/2 < \rho < 1$). Then

$$s_n(F) = O(n^{-1/\rho}L(n)).$$

Theorem 5. Let $A = F + T$ be a Volterra operator, where $F = F^*$ is a positive operator, and $T \in \mathfrak{S}_p$ for $p < 1/2$; then there exists and is finite the limit

$$\lim n^2 s_n(F).$$

Theorems 3 (for $1/2 < p < 1$), 4, and 5 are obtained by applying assertions A, B, and C to the function

$$f(\lambda) = g(\lambda^2),$$

where

$$g(\lambda) = \frac{1}{\det(I - i\lambda T R_\lambda(F))},$$

for which arguments analogous to those used in the proofs of Theorems 1 and 2 are valid. For $p > 1$, Theorem 3 follows from Theorem 1, and for $p = 1$, from a theorem of M. G. Krein ⁽⁴⁾.

Concerning Theorem 4, one may also note that

$$\overline{\lim} n^{1/\rho} s_n(F)[L(C)]^{-1} \leq C \overline{\lim} n^{1/\rho} s_n(T)[L(n)]^{-1},$$

where the constant C depends only on ρ and $L(r)$. M. G. Krein and the author conjecture that in Theorem 5 the condition $p < 1/2$ can be replaced by the condition $p = 1/2$. The validity of this conjecture was previously proved by M. G. Krein under the condition that the imaginary component of A be finite-dimensional.

We note that assertions A, B, and C can be generalized to meromorphic functions. In this way one obtains theorems on the dependence of the spectra of the components of an operator that is not assumed to be Volterra.

If $G = G^* \in \mathfrak{S}_\infty$, then by $\lambda_j^+(G)$ (respectively $\lambda_j^-(G)$) is denoted the j -th positive (respectively, the modulus of the negative) eigenvalue of the operator G , when they are numbered in decreasing order.

Consider all possible Volterra operators of the form $P_n + iQ$, where $Q = Q^*$, and P_n is a fixed n -dimensional orthogonal projector. Put

$$C_{n,m} = \sup_Q \sum_{j=1}^m \lambda_j^+(Q) \quad (m \geq 1), \quad C_{n,0} = 0.$$

It turns out that $C_{n,m} = C_{m,n}$. One can obtain a number of estimates for these absolute constants. The simplest of them follows directly from inequality (5) of ⁽¹⁾, namely

$$C_{n,m} \leq \frac{2}{\pi} n \sigma_m, \quad \sigma_m = \sum_{k=1}^m \frac{1}{2k-1}. \quad (5)$$

Theorem 6. If $A = G + iH$ is a Volterra operator, then

$$\max \left\{ \sum_{j=1}^n \lambda_j^+(G), \sum_{j=1}^n \lambda_j^-(G) \right\} \leq \sum_{k=1}^{\infty} (C_{n,k} - C_{n,k-1}) (\lambda_k^+(H) + \lambda_k^-(H))$$

$$(n = 1, 2, \dots),$$

and consequently

$$\max \left\{ \sum_{j=1}^n \lambda_j^+(G), \sum_{j=1}^n \lambda_j^-(G) \right\} \leq$$

$$\leq \frac{2}{\pi} \left[\sigma_n \sum_{k=1}^n (\lambda_k^+(H) + \lambda_k^-(H)) + n \sum_{k=n+1}^{\infty} \frac{\lambda_k^+(H) + \lambda_k^-(H)}{2k-1} \right] \quad (n = 1, 2, \dots). \quad (6)$$

Inequality (6) contains the inequalities indicated in (7, 9). It makes it possible to obtain a number of new results.

3. Let $\mathcal{H}(t)$ be some periodic, $\mathcal{H}(t+1) = \mathcal{H}(t)$, Hermitian function, integrable on the interval $(0, 1)$. Then the operator $A = G + iH$,

defined by the equality

$$(Af)(t) = 2i \int_t^1 \mathcal{H}(t-s)f(s) ds \quad (f \in L_2(0, 1)),$$

is a Volterra operator acting in the space $L_2(0, 1)$.

It is easy to verify that the systems of eigenvalues $\{\xi_j\}_{-\infty}^{\infty}$ and $\{\eta_j\}_{-\infty}^{\infty}$ of the operators H and G , under a suitable numbering, are related by the relations

$$\eta_k = \frac{1}{\pi} \sum_{l=-\infty}^{\infty} \frac{\xi_l}{l-k+\frac{1}{2}} \quad (k = 0, \pm 1, \dots). \quad (7)$$

In view of this, all the results obtained above, as well as the results of the papers ^{4,7}, make it possible to draw a number of conclusions concerning the transformation (7), which is a discrete analogue of the singular integral Hilbert transform ^{10,11}.

In particular, from Theorem 1 there follows the Riesz-Titchmarsh theorem on the boundedness of the transformation (7) in the space l_p ; moreover, the norm of this transformation, which coincides, as is known ^{10,11}, with the norm of the Hilbert transform in $L_p(-\infty, \infty)$, turns out not to exceed the exact constant C from Theorem 1.

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REFERENCES

- ¹ I. Ts. Gokhberg, M. G. Krein, DAN, **137**, No. 5 (1961).
- ² L. A. Sakhnovich, *Izv. Vyssh. uchebn. zaved., Matematika*, No. 4 (11) (1959).
- ³ I. Ts. Gokhberg, M. G. Krein, DAN, **128**, No. 2 (1959).
- ⁴ M. G. Krein, DAN, **130**, No. 3 (1960).
- ⁵ V. I. Macaev, DAN, **132**, No. 4 (1960).
- ⁶ B. Ya. Levin, *Distribution of Zeros of Entire Functions*, 1956.
- ⁷ I. Ts. Gokhberg, M. G. Krein, DAN, **139**, No. 4 (1961).
- ⁸ H. Weyl, *Proc. Nat. Acad. Sci. USA*, **35**, 408 (1949).
- ⁹ V. I. Macaev, DAN, **139**, No. 3 (1961).
- ¹⁰ M. Riesz, *Math. Zs.*, **27**, 2 (1927).
- ¹¹ E. C. Titchmarsh, *Math. Zs.*, **25**, 2 (1926).

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