



---

Soviet-era science, translated into English

# Mathematics

1961

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196101.54811>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Mathematics

I. S. Khara

## On One Method for Constructing Hermite' s Interpolation Formula and on Quadrature Formulas for Solving Boundary-Value Problems and Integral Equations

(Presented by Academician I. G. Petrovskii, 1 VII 1961)

1. In the Lagrange interpolation formula

$$f(x) = \sum_{k=1}^n l_k^{(n)}(x) f(x_k) + R(x) = \sum_{k=1}^n L_k^{(n)}(x) + R(x) \quad (1)$$

put

$$\left[ \prod_{k=1}^n (x - x_k) \right] : [(x - x_i)(x - x_{i+1})] = \omega_i(x); \quad x_{i+1} = x_i + h$$

and let  $h$  tend to zero. In all  $L_k^{(n)}(x)$ , except  $L_i^{(n)}(x)$  and  $L_{i+1}^{(n)}(x)$ , by continuity we put  $x_{i+1} = x_i$ , while in seeking

$$\lim_{h \rightarrow 0} (L_i^{(n)}(x) + L_{i+1}^{(n)}(x))$$

we proceed from

$$f(x_{i+1}) = \sum_{k=0}^{\infty} \frac{h^k}{k!} f^{(k)}(x_i).$$

The limiting formula will be an  $(n - 1)$ -point formula.

However, for convenience in what follows, we shall write the  $n$ -point formula

$$f(x) = \sum_{k=1}^n L_k^{(n,i(2))}(x) + R(x) = L_n^{(i(2))}(x) + R(x), \quad (2)$$

where the symbol  $i(2)$  means that the node  $x_i$  has been obtained by merging two nodes. We have

$$L_i^{(n,i(2))}(x) = f(x_i)\lambda_i(x) \left[ \left( \frac{1}{\lambda_i(x)} \right)_{x=x_i} + \left( \frac{1}{\lambda_i(x)} \right)'_{x=x_i} (x-x_i) \right] + \frac{(x-x_i)\lambda_i(x)}{1!} f'(x_i) \left( \frac{1}{\lambda_i(x)} \right)_{x=x_i}; \quad \lambda_i(x) = (x-x_{i+1})\omega_i(x).$$

By direct verification we are convinced that the values of the polynomial  $L_n^{(i(2))}(x)$ , whose uniqueness follows from the method of construction, at the nodes  $x_k$  are  $f(x_k)$ , and the value of the first derivative at the node  $x_i$  is  $f'(x_i)$ . Putting in (2)  $x_{i+1} = x_i + h$  and letting  $h$  tend to zero, we easily observe the law of formation of the coefficients at  $f^{(\nu)}(x_i)$ . We have

$$L_{i+1}^{(n,i(2))}(x) = \frac{(x-x_i)^2\omega_i(x)}{h^2\omega_i(x_i+h)} \sum_{k=0}^{\infty} \frac{h^k}{k!} f^{(k)}(x_i);$$

$$L_i^{(n,i(2))}(x) + L_{i+1}^{(n,i(2))}(x) = f(x_i)\omega_i(x)A_2(x) + f'(x_i)\frac{(x-x_i)\omega_i(x)}{1!}A_1(x) + f^{(2)}(x_i)\frac{(x-x_i)^2\omega_i(x)}{2!\omega_i(x_i+h)} + O(h);$$

$$A_2(x) = (x-x_i)(x-x_{i+1}) \left( \frac{1}{\lambda_i(x)} \right)'_{x=x_i} - \frac{x-x_{i+1}}{h\omega_i(x_i)} + \frac{(x-x_i)^2}{h^2\omega_i(x_i+h)} = \sum_{k=0}^2 A_2^{(k)}(h)(x-x_i)^k;$$

$$A_1(x) = -\frac{x-x_{i+1}}{h\omega_i(x_i)} + \frac{x-x_i}{h\omega_i(x_i+h)} = \sum_{k=0}^1 A_1^{(k)}(h)(x-x_i)^k;$$

$$A_2^{(2)}(h) = \left( \frac{1}{x-x_{i+1}} \frac{1}{\omega_i(x)} \right)'_{x=x_i} + \frac{1}{h^2\omega_i(x_i+h)} = -\frac{1}{h^2} \frac{1}{\omega_i(x_i)} - \frac{1}{h} \left( \frac{1}{\omega_i(x)} \right)'_{x=x_i} + \frac{1}{h^2} \sum_{k=0}^{\infty} \frac{h^k}{k!} \left( \frac{1}{\omega_i(x)} \right)_{x=x_i}^{(k)};$$

$$A_2^{(2)}(0) = \lim_{h \rightarrow 0} A_2^{(2)}(h) = \frac{1}{2!} \left( \frac{1}{\omega_i(x)} \right)''_{x=x_i};$$

$$A_2^{(1)}(0) = A_1^{(1)}(0) = \frac{1}{1!} \left( \frac{1}{\omega_i(x)} \right)'_{x=x_i};$$

$$A_2^{(0)}(0) = A_1^{(0)}(0) = \frac{1}{\omega_i(x_i)}.$$

In passing by induction from the formula with a  $(\nu - 1)$ -fold node  $x_i$  to the formula with a  $\nu$ -fold node  $x_i$ , the computation of the coefficients of  $(x - x_i)^k$ , analogous only to those already computed, is carried out approximately as the coefficient  $A_2^{(2)}(0)$  was computed. We do not give a detailed writing of Hermite's formula, since it is well known.

II. Writing Hermite's formula for the interval  $[-b, b]$  in the form

$$\psi(x) = \sum_{j=1}^n \sum_{i=0}^{\alpha_j-1} b^i H_{ij} \left( \frac{x}{b} \right) \psi^{(i)}(x_j) + R_H, \quad (3)$$

we introduce for consideration the quadrature formula

$$\int_{-b}^t \int_{-b}^x \cdots \int_{-b}^x \psi(x) dx^\nu = b^\nu \sum_{j=1}^n \sum_{i=0}^{\alpha_j-1} b^i A_{ij} \psi^{(i)}(x_j) + R, \quad (4)$$

$$A_{ij} = b^{-\nu} \int_{-b}^t \int_{-b}^x \cdots \int_{-b}^x H_{ij} \left( \frac{x}{b} \right) dx^\nu. \quad (5)$$

For solving many types of integral equations and boundary-value problems (the existence of solutions—unique or nonunique—is assumed), it is sufficient to have a small collection (several printed sheets) of tables of coefficients  $A_{ij}$  and tables (with step  $h = 0.05b$ ) of the functions  $H_{ij}(\frac{x}{b})$  and their first derivatives. A brief list of tables of recommended coefficients is given below.

Types of nodes: 1)  $x_j = \frac{2j - n - 1}{n}b$ ; 2)  $x_j = \frac{2j - n - 1}{n - 1}b$ . For nodes of the first type:  $\nu = 1$ ;  $t = b$ ;  $A_{ij} = A_{ij}^{(n)}[\alpha_1, \dots, \alpha_n]$ ;  $R = R^{(n)}[\alpha_1, \dots, \alpha_n]$ . For nodes of the second type:  $\nu = 1, 2, 3, 4$ ;  $t = x_k$  ( $k = 2, \dots, n$ );  $A_{ij}^{(\nu, k, n)}[\alpha_1, \dots, \alpha_n]$ ;  $R^{(\nu, k, n)}[\alpha_1, \dots, \alpha_n]$ . We denote the tables by  $T_k$ .  $T_1$ :  $A_{ij}^{(3)}[2, 2, 2]$ ;  $T_2$ :  $A_{ij}^{(2)}[\alpha, \alpha]$ ;  $T_3$ :  $A_{ij}^{(\nu, 2, 2)}[\alpha, \alpha]$ ;  $T_4$ :  $A_{ij}^{(\nu, k, 3)}[\alpha, \alpha, \alpha]$ ;  $T_5$ :  $A_{ij}^{(\nu, k, 3)}[\alpha, 1, \alpha]$ . For tables  $T_2$ – $T_5$ ,  $\alpha = 2, 3, 4$ . Expressions for the remainder terms corresponding to tables  $T_1$ – $T_5$ , in terms of values of the corresponding derivative of the function  $\psi(x)$  at one point of the interval  $[-b, b]$

in some cases are obtained by direct application of the mean value theorem, and in others by means of the method proposed in ([1], p. 243). The coefficients  $A_{ij}$  are easily found from systems of equations. Thus, for example, in order

to determine the 6 coefficients  $A_{ij}^{(v,2,2)}[3,3]$  in (4), we put  $\psi(x) = x^m$  ( $m = 0, 1, \dots, 5$ ). The system of 6 equations splits into two systems.  $A_{01} + A_{02}$ ,  $A_{11} - A_{12}$ , and  $A_{21} + A_{22}$  are determined from one system, while  $A_{01} - A_{02}$ ,  $A_{11} + A_{12}$ , and  $A_{21} - A_{22}$  are determined from the other. As is known, when nonlinear integral equations are solved by ordinary quadrature formulas (i.e., quadratures with simple nodes), systems of equations (algebraic or transcendental) of a more or less general form are obtained, the solution of which, with a considerable number of unknowns, requires a large expenditure of labor. When formula (4) is applied to the solution of integral equations, it proves possible to obtain systems of equations of a special form; the next section is devoted to their description and to an algorithm for solving them.

III. Let, for the system of equations

$$x_{ij} = F_{ij}^{(1)} + F_{ij}^{(2)} \quad (i = 1, 2; j = 1, 2, \dots, n_i) \quad (6)$$

the following conditions be satisfied with respect to two groups of unknowns  $x_{ij}$  ( $i$  is the number of the group,  $j$  the number of the unknown in the group): 1)  $F_{ij}^{(1)}$  depends only on the unknowns of the first group; 2)  $F_{ij}^{(2)}$ , which in the general case depend on the unknowns of both groups (in linear systems they depend only on the unknowns of the second group), are small quantities, substantially less than unity in absolute value. We shall agree to call  $x_{1j}$  the principal unknowns, and  $x_{2j}$  the additional unknowns. In systems (6) arising in the solution of integral equations, the principal unknowns are, as a rule, the values of the desired solution, and the additional unknowns are the values of the derivatives of the desired solution.

**Construction of the first approximation  $x_{ij}^{(1)}$ .** First we find  $x_{1j}^{(1)}$  as the solution of the system of equations  $x_{1j}^{(1)} = F_{1j}^{(1)}$ , and then, having computed from the found  $x_{1j}^{(1)}$  the values  $F_{2j}^{(1)}$ , we determine  $x_{2j}^{(1)}$  by means of the equalities  $x_{2j}^{(1)} = F_{2j}^{(1)}$ .

**Construction of the  $k$ -th ( $k > 1$ ) approximation  $x_{ij}^{(k)}$ .** Having found, from the results of the  $(k - 1)$ -st approximation, the values  $F_{ij}^{(2)}$  for all  $i$  and  $j$ , we first find  $x_{1j}^{(k)}$  as the solution of the system  $x_{1j}^{(k)} = F_{1j}^{(1)} + F_{1j}^{(2)}$ , and then, having computed from the found  $x_{1j}^{(k)}$  the values  $F_{2j}^{(1)}$ , we determine  $x_{2j}^{(k)}$  by means of the equalities  $x_{2j}^{(k)} = F_{2j}^{(1)} + F_{2j}^{(2)}$ .

For ease of reference we shall agree to call system (6) an  $F$ -system, and the scheme of solution described above scheme (algorithm)  $D$ . In order to compare algorithm  $D$  and the simple iteration algorithm to some extent, suppose that system (6) is a linear  $F$ -system. Writing system (6) in matrix form  $X = AX + B$ , represent the matrix  $A$  in the form of a square matrix of order two  $A = (A_{ik})$ , whose elements  $A_{ik}$  are matrices of the form

$$F_{ij}^{(k)} = \sum_{\nu=1}^{n_k} a_{ij}^{(k,\nu)} x_{k\nu}.$$

As a rule, for linear systems the quantities  $F_{ij}^{(2)}$  are small (the second property of  $F$ -systems) when the  $a_{ij}^{(2,\nu)}$  are small. Since for  $k \geq i$  all the  $A_{ij}$ , with the exception of  $A_{11}$ , are small (have small norms), only one element,  $A_{11}$ , can hinder the rapid decrease of the powers of the matrix  $A$ , and consequently the rapid convergence of the simple iteration algorithm. Solving the system  $X = AX + B$  by scheme  $D$  is equivalent, roughly speaking, to the application of the simple iteration algorithm preceded by the determination of the  $n_1$  principal unknowns from the first  $n_1$  equations of the system in terms of the additional unknowns, as a result of which, for the new system, the element  $A_{11}$  becomes equal to zero. Rapid

the convergence of algorithm  $D$  was confirmed in the solution of a number of nonlinear integral equations.

IV. Formula (4) allows the solution of boundary-value problems to be reduced to the solution of systems of equations with a small number of unknowns.

Suppose it is required to solve the boundary-value problem

$$y'' = f(x, y, y'), \quad y(0) = \alpha_0, \quad y(1) = \alpha_1 \quad (7)$$

using the coefficients  $A_{ij}^{(v,k,3)}$  [3, 3, 3]. The system of equations for the 9 unknowns  $y_j^{(i)} = (d^i y / dx^i)_{x=x_j}$ , where  $x_j = 0.5(j-1)$ , naturally splits into two groups. The equations  $y_j'' = f(x_j, y_j, y_j')$ , together with the boundary conditions, form the first group; the second group of equations is obtained by carrying out repeated integration of equation (7) from 0 to  $x$  and replacing the integrals for  $x = x_2$  and  $x = x_3$  by finite sums using the coefficients  $A_{ij}^{(v,k,3)}$  [3, 3, 3]. The system of equations with 9 unknowns described above is easily reduced, by means of the equations of the first group, to a system with 4 unknowns. The boundary-value problem (7), by means of the tables  $T_3$ , is reduced to a system with 2 unknowns, and if equation (7) does not contain  $y'$ , then by means of the tables  $T_5$ —to a system with 3 unknowns.

Kharkov Polytechnic Institute  
named after V. I. Lenin

Received  
28 VI 1961

## References Cited

1. I. S. Berezin, N. P. Zhidkov, *Methods of Computation*, 1, 1960.

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*