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Abstract

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ON REGIONS OF VARIATION OF SYSTEMS OF COEFFICIENTS OF FUNCTIONS REPRESENTABLE AS A SUM OF STIELTJES INTEGRALS

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The paper investigates the regions of variation of arbitrary finite systems of coefficients of Laurent expansions of functions representable in a finitely connected circular domain as a sum of Stieltjes integrals and, in particular, of functions typically real in a disk and in a circular annulus. Along with theorems of a general character, some concrete results are obtained. In particular, the region of variation of the system $\{f(z_0), c_2\}$ is found, where z_0 is a fixed point of the unit disk E, and $f(z) = z + c_2 z^2 + \cdots$ runs through the class of functions typically real in E.

§ 1. Let G be a finitely connected circular domain of the z-plane, bounded for simplicity, and let F be the class of all functions $f(z)$ representable in G by the formula

$$f(z) = \sum_{k=1}^{m} \int_a^b g_k(z,t)\, d\mu_k(t),$$

where $g_k(z,t)$ are functions fixed for the class F, regular in G with respect to z, continuous on $[a,b]$ with respect to t, and uniformly bounded inside G for all $t \in [a,b]$, while $\mu_k(t)$ are arbitrary nondecreasing functions on $[a,b]$ with

$$\int_a^b d\mu_k(t) = 1, \qquad k = 1, \dots, m.$$

Let arbitrary distinct points z_ν, $\nu = 1, \dots, s$, be given, each of which is either a point of the domain G or the center of one of the circles bounding this domain. Then each function $f(z)$ of the class F and the functions $g_k(z,t)$, $k = 1, \dots, m$, are expanded in series in powers of $z - z_\nu$, in each disk and circular annulus with center at the point z_ν lying in G, $\nu = 1, \dots, s$. Let, for the given function $f(z)$ of the class F and for the functions $g_k(z,t)$, these expansions in one of the indicated subdomains G be

$$f(z) = \sum_{n=-\infty}^{\infty} c_n^{(\nu)} (z - z_\nu)^n, \qquad \nu = 1, \dots, s, \tag{1}$$

$$g_k(z,t)=\sum_{n=-\infty}^{\infty} a_{n,k}^{(\nu)}(t)(z-z_\nu)^n,\qquad \nu=1,\dots,s,\quad k=1,\dots,m,\quad t\in[a,b].$$

Choose from the entire set of coefficients of the expansions (1) N ($N \geqslant 1$) arbitrary coefficients, arrange them in a fixed order, and, writing the resulting system of coefficients in the form

$$C_1,\dots,C_N, \tag{2}$$

consider it as a point $P=\mathcal{F}[f]=\{C_1,\dots,C_N\}$ of the N-dimensional complex space* $\mathcal{R}_N$. In $\mathcal{R}_N$ the points $P=\mathcal{F}[g_k(z,t)]$ are also defined,

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* Here and below Euclidean spaces are meant.

which we shall write in the form

$$\{A_1^{(k)}(t),\dots,A_N^{(k)}(t)\},\qquad k=1,\dots,m,\qquad t\in[a,b].$$

By the **range of variation** of the system (2) of coefficients of functions of the class F we shall mean the set $\mathcal{E}(F)$ of points $P=\mathfrak{P}[f]$ of the space $\mathcal{R}_N$, obtained under the condition that the points z_ν, $\nu=1,\dots,s$, are fixed, while the function f ranges over the whole class F. Functions realizing, in the class F, boundary points of the domain $\mathcal{E}(F)$ will be called **boundary functions** of this domain. By $M=\sum_{k=1}^{l} M_k$ we shall denote the geometric sum of the sets $M_1,\dots,M_l$ of the space $\mathcal{R}_N$; by $\mathcal{K}(U)$, where U is any closed bounded set in $\mathcal{R}_N$, the convex hull of the set U. Put

$$U_k=\bigcup_{a\le t\le b}\mathfrak{P}[g_k(z,t)]=\bigcup_{a\le t\le b}\{A_1^{(k)}(t),\dots,A_N^{(k)}(t)\},\qquad k=1,\dots,m.$$

Theorem 1. $\mathcal{E}(F)\equiv\mathcal{K}\left(\sum_{k=1}^{m}U_k\right)$.

The proof is carried out by using the simplest results of the theory of convex bodies.

§ 2. Let T be the class of all functions of the form $f(z)=z+\sum_{n=2}^{\infty}c_n z^n$, regular and typically real in the disk $|z|<1$, i.e. satisfying there the condition $\operatorname{Im} z\cdot\operatorname{Im} f(z)>0$ for $\operatorname{Im} z\neq 0$. Let integers $m_1,\dots,m_N$ be given, for which $2\le m_1<\dots<m_N$, $N\ge 1$. Put $\rho_n(t)=\sin n\theta/\sin\theta$, $t=\cos\theta$, $n=1,2,\dots$.

Theorem 2. The range of variation $\mathcal{E}(T)$ of the system $\{c_{m_1},\dots,c_{m_N}\}\equiv\{x_1,\dots,x_N\}$ of coefficients of functions of the class T is the smallest convex body of N-dimensional real space containing the arc $x_l=\rho_{m_l}(t)$, $l=1,\dots,N$, $-1\le t\le 1$. The boundary functions of the body $(\mathcal{E}(T))$ are only functions of the form

$$f(z) = z \sum_{j=1}^{p} \frac{\lambda_j}{1 - 2t_j z + z^2},$$

where $1 \leq p \leq \left[\frac{m_N + 1}{2}\right]$; $t_j \in [-1, 1]$, $j = 1, \dots, p$; $t_j \neq t_{j'}$ for $j \neq j'$; $\lambda_j > 0$; $\sum_{j=1}^{p} \lambda_j = 1$, and where the upper bound for p cannot be replaced by a smaller one.

With the help of this theorem it is not difficult to find the ranges of variation of certain systems of two Maclaurin coefficients of functions of the class T, the sharp estimates of each of these two coefficients in terms of the other, and all extremal functions for these estimates. We note that the question of the range of variation of coefficient systems of the form $\{c_2, c_3, \dots, c_{N+1}\}$ in the class T was completely solved by Rogozinskii ([1]).

Consider for the class T the range of variation of the system $\{f(z_0), c_2\}$, where z_0, $0 < |z_0| < 1$, is fixed. We shall regard this system as a point $\{w, x_3\} \equiv \{x_1, x_2, x_3\}$ of three-dimensional real space, where $f(z_0) = w = x_1 + ix_2$, $c_2 = x_3$.

Theorem 3. For $\operatorname{Im} z_0 \neq 0$ the range of variation $\mathcal{E}(T)$ of the system $\{f(z_0), c_2\}$ in the class T is a body of three-dimensional real space,

bounded by two conical surfaces:

$$\left| w - \frac{z_0}{(1 \mp z_0)^2} \right|^2 \operatorname{Im}\left\{ z_0 + \frac{1}{z_0} \right\} = (x_3 \mp 2) \operatorname{Im}\left\{ \frac{z_0}{(1 \mp z_0)^2} \overline{w} \right\} \qquad (w = x_1 + ix_2).$$

For $\operatorname{Im} z_0 = 0$, $\mathcal{E}(T)$ is a plane domain bounded by an arc of the hyperbola

$$x_3 = z_0 + \frac{1}{z_0} - \frac{1}{x_1}, \qquad \frac{z_0}{(1 + z_0)^2} \leq x_1 \leq \frac{z_0}{(1 - z_0)^2},$$

and by the chord contracting its endpoints.

To each boundary point $\{w, c_2\}$ of the body $\mathcal{E}(T)$ there corresponds a unique boundary function

$$f(z) = \lambda \frac{z}{(1 \mp z)^2} + (1 - \lambda) \frac{z}{1 - 2tz + z^2},$$

where the values of λ and t form the solution, belonging to the domain $0 \leq \lambda \leq 1$, $-1 \leq t \leq 1$, of the system of equations:

$$\lambda \frac{z_0}{(1 \mp z_0)^2} + (1 - \lambda) \frac{z_0}{1 - 2tz_0 + z_0^2} = w, \qquad \mp 2\lambda + 2(1 - \lambda)t = c_2,$$

and where the upper signs correspond to a point of the upper boundary surface, and the lower signs to a point of the lower boundary surface.

For $\operatorname{Im} z_0 = 0$, to each point $\{x_1, c_2\}$ of the arc of the hyperbola bounding the domain $\mathcal{E}(T)$ there corresponds a unique boundary function

$$f(z) = \frac{z}{1 - c_2 z + z^2},$$

and to a point of the chord contracting it—the function

$$f(z) = \frac{z(1 + c_2 z + z^2)}{(1 - z^2)^2}.$$

The proof of the theorem is based on verifying that the convex hull of the arc

$$w = \frac{z_0}{1 - 2t z_0 + z_0^2}, \qquad x_3 = 2t, \qquad -1 \le t \le 1,$$

is a body bounded by two conical surfaces, whose directrix is this arc and whose vertices are its endpoints.

Denote by $T(c_2)$ the subclass of all functions

$$f(z) = z + c_2 z^2 + \cdots$$

from T with fixed coefficient c_2.

Corollary. Let $\mathcal{E}(T(c_2))$ be the region of variability of $f(z_0)$ under the condition that z_0, $0 < |z_0| < 1$, is fixed, and the function $f(z)$ ranges over the class $T(c_2)$. For $\operatorname{Im} z_0 \neq 0$, the domain $\mathcal{E}(T(c_2))$ is a circular lune lying in the disk

$$\left| w + \frac{i}{2\eta} \right| \le \frac{1}{2|\eta|}$$

and bounded by two circles*:

$$\left| w - \left(1 - \frac{c_2 \mp 2}{2\eta}\, i\right) \frac{z_0}{(1 \mp z_0)^2} \right| = \frac{2 \mp c_2}{2|\eta|} \left| \frac{z_0}{(1 \mp z_0)^2} \right|, \qquad \eta = \operatorname{Im}\left\{ z_0 + \frac{1}{z_0} \right\}.$$

For $\operatorname{Im} z_0 = 0$, the domain $\mathcal{E}(T(c_2))$ is a segment of the real axis joining the points

$$\frac{z_0}{1 - c_2 z_0 + z_0^2} \quad \text{and} \quad \frac{z_0(1 + c_2 z_0 + z_0^2)}{(1 - z_0^2)^2}.$$

§ 3. Let T_q be the class of all functions of the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n,$$

regular and typically real in the annulus $q < |z| < 1,\ q > 0$, i.e., satisfying in it the condition

$$\operatorname{Im} z \cdot \operatorname{Im} f(z) > 0 \quad \text{for} \quad \operatorname{Im} z \neq 0.$$

Denote by $T_q(c_{-1}, c_1)$ the subclass of functions from T_q with fixed coefficients c_{-1} and c_1. Put

$$S_q(z,t) = \sum_{\nu=-\infty}^{\infty} \frac{q^{2\nu} z}{1 - 2tq^{2\nu} z + q^{4\nu} z^2}$$

and consider the polynomials

$$p_n(t) = \frac{\sin n\theta}{\sin\theta}, \qquad t = \cos\theta,$$

for arbitrary $n = \pm 1, \pm 2, \ldots$. Let arbitrary integers $m_1, \ldots, m_N$, distinct from 0 and ± 1, be given, for which

$$m_1 < \cdots < m_N, \qquad N \geq 1.$$

* The upper signs correspond to one of the circles, the lower signs to the other.

Theorem 4. The range $\mathcal{E}$ of the system $\{c_{m_1}, \ldots, c_{m_N}\} = \{x_1, \ldots, x_N\}$ of coefficients of functions of the class $T_q(c_{-1}, c_1)$ is the convex hull of the geometric sum of two arcs of an N-dimensional real space:

$$x_l = (c_1 - c_{-1}) \frac{p_{m_l}(t)}{1 - q^{2m_l}}, \qquad l = 1, \ldots, N, \quad -1 \leq t \leq 1,$$

and

$$x_l = -(c_1 q - c_{-1} q^{-1}) \frac{q^{m_l} p_{m_l}(t)}{1 - q^{2m_l}}, \qquad l = 1, \ldots, N, \quad -1 \leq t \leq 1.$$

If $|m_j| \neq |m_{j'}|$ for $j \neq j'$, the boundary functions of the domain $\mathcal{E}$ are only functions of the form

$$f(z) = (c_1 - c_{-1}) \sum_{j=1}^{p_1} \lambda_j^{(1)} S_q(z, t_j^{(1)}) - (c_1 q - c_{-1} q^{-1}) \sum_{j=1}^{p_2} \lambda_j^{(2)} S_q(qz^{-1}, t_j^{(2)}) + c_0,$$

where

$$1 \leq p_k \leq \left[\frac{m+1}{2}\right]; \qquad m = \max(|m_1|, |m_N|); \qquad \lambda_j^{(k)} > 0, \quad \sum_{j=1}^{p_k} \lambda_j^{(k)} = 1;$$

$$t_j^{(k)} \in [-1, 1], \qquad t_j^{(k)} \neq t_{j'}^{(k)} \text{ for } j \neq j', \quad k = 1, 2,$$

c_0 is a real constant, and the upper bound for p_k cannot be replaced by a smaller one.

Let us note that the domain $\mathcal{E}$ considered in this theorem is not always a body in N-dimensional real space, and that, when the condition $|m_j| \neq |m_{j'}|$, $j \neq j'$, is not fulfilled, the boundary functions of the domain $\mathcal{E}$ are not always only functions of the form indicated in the theorem.

Theorem 4 makes it possible to find the ranges of some of the simplest systems of coefficients of functions of the class $T_q(c_{-1}, c_1)$, and, in particular, the range of the system $\{c_{-n}, c_n\}$ for any $n \geq 2$. This gives a strengthening of the known [2] estimates of the coefficients of functions of the class $T_q(c_{-1}, c_1)$.

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[2] Y. Komatu, *Kodai Math. Sem. Rep.*, **9**, No. 1 (1957).

Note: Figure translations are in progress. See original paper for figures.