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Abstract

Full Text

MATHEMATICS

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A GENERALIZATION OF THE STRUCTURAL FORMULA FOR CONVEX UNIVALENT FUNCTIONS TO THE CASE OF A MULTIPLY CONNECTED CIRCULAR DOMAIN

(Presented by Academician V. I. Smirnov, 1 VIII 1960)

The well-known structural formula for the class of convex univalent functions regular in the disk $|z| < 1$ was generalized by V. A. Zmorovich to the case of the annulus $q < |z| < 1$ ⁽¹⁾. In the present note a further generalization of this formula is given for the case of a multiply connected circular domain—the structural formula for the class $B(D_{n+1})$ of functions $f(z)$, regular and univalent in the domain D_{n+1} , bounded from the outside by the circle $|z - a_0| = r_0$ and from the inside by the circles $|z - a_k| = r_k$; ($k = 1, 2, \dots, n$); $|a_k - a_l| > r_k + r_l$; $k, l = 1, 2, \dots, n$; $k \neq l$, mapping D_{n+1} onto domains bounded by $n + 1$ convex lines.

The solution is given in constructive form, but not for all domains D_{n+1} , only for those satisfying certain conditions characterizing the convergence of the series involved in the formation of the structural formula. In its construction a method was used that had been employed by G. M. Goluzin in solving Dirichlet and Neumann type problems ⁽²⁾. We note that in the work of L. O. Dunduchenko and S. A. Kasyanyuk ⁽³⁾ a construction is carried out of a structural formula for the class of functions mapping the z -plane with n disks $|c_k - z| \leq R_k$ ($k = 1, 2, \dots, n$) removed onto domains of bounded boundary rotation in the sense of Paatero ⁽⁴⁾, in particular onto domains bounded by convex lines and polygons. However, that work contains an error, and as a result the structural formula turns out to be incorrect. It appears impossible to solve this problem by the method used by the authors mentioned.

Denote by C_m the circle $|z - a_m| = r_m$ ($m = 0, 1, \dots, n$). Let C_m^ρ be a circle concentric with C_m , lying wholly in D_{n+1} and at distance ρ from C_m .

Consider the auxiliary class $K(D_{n+1})$ of functions $\varphi(z)$ satisfying the following conditions:

- 1) $\varphi(z)$ is regular in D_{n+1} and can be represented in the form

$$\begin{aligned} \varphi(z) &= b_0^{(0)} + b_1^{(0)}(z - a_0) + b_2^{(0)}(z - a_0)^2 + \dots \\ &\dots + \sum_{m=1}^n \left[\frac{b_2^{(m)}}{(z - a_m)^2} + \frac{b_3^{(m)}}{(z - a_m)^3} + \dots \right]. \end{aligned} \quad (1)$$

2) On the circles C_m^ρ ($m = 0, 1, \dots, n$), for sufficiently small ρ ,

$$\int_{-\pi}^{\pi} \alpha_{m,\rho}^-(\theta) d\theta < \varepsilon \quad (2)$$

whatever $\varepsilon > 0$ may be. Here $\theta = \arg(z - a_m)$; $z \in C_m^\rho$;

$$\alpha_{m,\rho}^-(\theta) = \begin{cases} -\operatorname{Re}[1 + (z - a_m)\varphi(z)], & \text{if } \operatorname{Re}[1 + (z - a_m)\varphi(z)] \leq 0, \\ 0, & \text{if } \operatorname{Re}[1 + (z - a_m)\varphi(z)] > 0. \end{cases}$$

Theorem 1. *The structural formula*

$$\varphi(z) = \frac{1}{2\pi} \sum_{k=0}^n \int_{-\pi}^{\pi} \Phi_k(z, \theta) d\mu_k(\theta), \quad (3)$$

where $\mu_k(\theta)$ are nondecreasing on $[-\pi, \pi]$ and

$$\int_{-\pi}^{\pi} d\mu_k(\theta) = 2\pi \quad (k = 0, 1, 2, \dots, n),$$

is a necessary and sufficient condition for the function $\varphi(z)$ to belong to the class $k(D_{n+1})$. Here

$$\Phi_k(z, \theta) = \sum_{\nu=1}^{\infty} \left[h_{\nu 0k}(z, \theta) + \sum_{m=1}^n \frac{h_{\nu mk}(z, \theta)}{(z - a_m)^2} \right] + \frac{F_k(z, \theta)}{(z - a_k)^2} \quad (k = 1, 2, \dots, n);$$

$$\Phi_0(z, \theta) = \sum_{\nu=1}^{\infty} \left[h_{\nu 00}(z, \theta) + \sum_{m=1}^n \frac{h_{\nu m0}(z, \theta)}{(z - a_m)^2} \right] + F_0(z, \theta); \quad (4)$$

$$F_0(z, \theta) = \frac{2e^{-i\theta}}{r_0 - (z - a_0)e^{-i\theta}}; \quad F_k(z, \theta) = \frac{-2r_k(z - a_k)}{r_k - (z - a_k)e^{-i\theta}}; \quad (5)$$

for ν odd,

$$\begin{aligned}
 h_{\nu mk}(z, \theta) &= \sum_{k_{\nu-1}, \dots, k_1} B_{mk_{\nu-1}}(z) \overline{B_{k_{\nu-1}k_{\nu-2}}([m, z])} \dots \\
 &\dots B_{k_2k}([k_2, \dots, m, z]) \overline{F_k([k_1, k_2, \dots, k_{\nu-1}, m, z], \theta)}; \quad (6)
 \end{aligned}$$

for ν even,

$$\begin{aligned}
 h_{\nu mk}(z, \theta) &= \sum_{k_{\nu-1}, \dots, k_1} B_{mk_{\nu-1}}(z) \overline{B_{k_{\nu-1}k_{\nu-2}}([m, z])} \dots \\
 &\dots \overline{B_{k_1k}([k_2, \dots, m, z])} F_k([k_1, k_2, \dots, k_{\nu-1}m, z], \theta).
 \end{aligned}$$

In formulas (6), $k, m = 0, 1, \dots, n$; the summation over $k_1, \dots, k_{\nu-1}$ is carried out from 0 to n ;

$$\begin{aligned}
 B_{mk}(z) &= \frac{-r_m^2}{([m, z] - \bar{a}_k)^2}, \quad m, k = 1, 2, \dots, n; \quad m \neq k; \\
 B_{0k}(z) &= \frac{-r_0^2}{[(\bar{a}_0 - \bar{a}_k)(z - a_0) + r_0^2]^2}, \quad k = 1, 2, \dots, n; \quad (7) \\
 B_{m0}(z) &= -r_m^2, \quad m = 1, 2, \dots, n; \\
 B_{kk}(z) &= 0, \quad k = 0, 1, \dots, n;
 \end{aligned}$$

for ν odd,

$$\begin{aligned}
 [m, z] &= a_m + \frac{r_m^2}{z - \bar{a}_m}; \\
 [k_1, k_2, \dots, k_{\nu-1}, k_\nu, z] &= \\
 &= a_{k_1} + \frac{r_{k_1}^2}{-a_{k_1} + \bar{a}_{k_2} + \frac{r_{k_2}^2}{-a_{k_2} + \bar{a}_{k_3} + \dots + \frac{r_{k_{\nu-1}}^2}{-a_{k_{\nu-1}} + \bar{a}_\nu + \frac{r_\nu^2}{-\bar{a}_\nu + z}}}};
 \end{aligned}$$

for even ν

$$\begin{aligned}
 & [k_1, k_2, \dots, k_{\nu-1}, k_\nu, z] = \\
 & = a_{k_1} + \frac{r_{k_1}^2}{-a_{k_1} + a_{k_2} + \frac{r_{k_2}^2}{-a_{k_2} + a_{k_3} + \dots + \frac{r_{k_{\nu-1}}^2}{-a_{k_{\nu-1}} + a_\nu + \frac{r_\nu^2}{-a_\nu + z}}}}.
 \end{aligned}$$

Let

$$\begin{aligned}
 \delta_m &= \left[1 + \sum_{\substack{s=1 \\ s \neq m}}^n \frac{1}{|a_m - a_s| - r_m} \right] r_m^2, \quad m = 1, 2, \dots, n; \\
 \delta_0 &= \sum_{s=1}^n \frac{1}{[r_0 - |a_0 - a_s|]^2}.
 \end{aligned}$$

Theorem 2. If the conditions

$$\delta_m < 1, \quad m = 0, 1, 2, \dots, n, \tag{8}$$

are satisfied, then the series (4) converge absolutely and uniformly in the domain D_{n+1} .

These conditions, roughly speaking, will be satisfied when r_m ($m = 1, 2, \dots, n$) are sufficiently small, and r_0 is sufficiently large.

Main theorem. The structural formula

$$f(z) = \int_{z_0}^z \exp \left[\int_{z_1}^z \varphi(z) dz \right] dz, \tag{9}$$

where z_0, z_1 are fixed points of D_{n+1} , $\varphi(z) \in K(D_{n+1})$, is a necessary and sufficient condition for the function $f(z)$ to belong to the class $B(D_{n+1})$ under the normalization $f(z_0) = 0$, $f'(z_1) = 1$. The function $\varphi(z)$ is chosen so that all residues of the integrand

$$\exp \left[\int_{z_1}^z \varphi(z) dz \right]$$

at the points a_k ($k = 1, 2, \dots, n$) are equal to zero.

Corollary 1. If it is assumed that the functions $\mu_k(\theta)$ entering formula (9) are step functions with points of discontinuity θ_{kj_k} ($k = 0, 1, \dots, n$; $j_k = 1, 2, \dots, \nu_k$; $-\pi < \theta_{k1} < \theta_{k2} < \dots < \theta_{k\nu_k} < \pi$) and jumps at these points $2\pi\sigma_{kj_k}$ ($\sigma_{kj_k} > 0$; $\sum_{j_k=1}^{\nu_k} \sigma_{kj_k} = 1$), then formula (9) gives regular, univalent in D_{n+1} functions $f(z)$, mapping D_{n+1} onto domains bounded by convex polygons with vertices $f(r_k e^{i\theta_{kj_k}})$ and with interior angles at them equal to $\pi(1 - 2\sigma_{kj_k})$.

The mapping function in this case has the form (without normalization)

$$f(z) = c \int_{z_0}^z \prod_{k=0}^n \prod_{j_k=1}^{\nu_k} \Psi_{kj_k}(z) dz + c_1, \quad (10)$$

where

$$\Psi_{kj_k}(z) = \exp \left[\sigma_{kj_k} \int_{z_1}^z \Phi_k(z, \theta_{kj_k}) dz \right]. \quad (11)$$

Corollary 2. If in formula (11) the σ_{kj_k} are of arbitrary sign, but remain subject to the conditions

$$\sum_{j_k=1}^{\nu_k} \sigma_{kj_k} = 1 \quad (k = 0, 1, \dots, n),$$

then formula (10) will give functions mapping D_{n+1} onto domains bounded by polygons (not necessarily convex or even univalent), and formula (10) ensures that $f'(z) \neq 0$, i.e., the absence of branch points in the image. However, in this case the mapping will, generally speaking, be non-univalent. Formula (10) in this case represents a generalization of the well-known Christoffel-Schwarz formula to the case of a multiply connected circular domain. In particular, for $n = 2$, in the case of a concentric circular annulus we obtain the Tiri-Akhiezer-Goluzin formula⁽¹⁾. For $n = 1$ we have the Christoffel-Schwarz formula for the disk.

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Note: Figure translations are in progress. See original paper for figures.

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