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Abstract

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SINGULAR INTEGRAL EQUATIONS IN CLASSES OF LIPSCHITZ FUNCTIONS

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In the present note we retain the terminology and notation of the papers ([1,2]).

1°. The purpose of the present note is to establish sufficient conditions under which the solution (if it exists) of the multidimensional singular integral equation

$$a(x)u(x) + \int_{E_m} \frac{f(x,\theta)}{r^m} u(y)\,dy = g(x),$$

$$r = |y - x|, \qquad \theta = \frac{y-x}{r}, \tag{1}$$

satisfies a Lipschitz condition with some positive exponent. We assume that the free term satisfies the following condition: the product

$$\left(\frac{1+|x|^2}{2}\right)^{m/2} g(x) \in \mathrm{Lip}_\alpha(\Sigma).$$

Here Σ is the sphere into which the Euclidean space E_m is transformed under stereographic projection. From the external condition it follows, in particular, that $g(x) \in L_2(E_m)$. The matter is reduced to imposing sufficient conditions on $a(x)$ and $f(x,\theta)$. As usual, we assume that the symbol of equation (1) nowhere vanishes.

The problem posed here was considered by Giraud ([3]), who, however, investigated a very special class of equations whose characteristics are spherical functions of the first order.

Below we shall use the following, essentially well-known, lemma.

Lemma. Let Ω be a finite domain of the space E_m, and let in this domain the function $A(x,y)$ satisfy the inequalities

$$|A(x,y)| \le C, \qquad |A(x+h,y) - A(x,y)| \le N|h|^\alpha, \qquad 0 < \alpha < 1, \tag{2}$$

where C, N, α are constants. Then the integral operator with weak singularity

$$v(x) = \int_{\Omega} \frac{A(x,y)}{r^{\lambda}} u(y)\, dy, \qquad 0 \le \lambda < m, \tag{3}$$

maps every bounded function $u(x)$ into a function $v(x) \in \mathrm{Lip}_{\beta}(\Omega)$, where $\beta = \min(\alpha, m-\lambda)$.

2°. We assume that $a(x)$ and $f(x,\theta)$ in equation (1) satisfy the following requirements:

a) $a(x) \in C^{(1)}(\Sigma)$.

b) $f(x,\theta) \in \hat{W}_2^{(l)}(S)$, $l \ge m+2$.

c) Let $\omega(x,r,\theta)$ be an arbitrary function of its arguments. Denote by $\partial'\omega/\partial x_j$ the derivative computed under the assumption that

r and θ do not depend on x, and by $\partial''\omega/\partial x_j$ we mean the derivative computed under the assumption that only r and θ depend on x, so that

$$\frac{\partial \omega}{\partial x_j} = \frac{\partial' \omega}{\partial x_j} + \frac{\partial'' \omega}{\partial x_j}.$$

It is easy to see that a formula of the form

$$\frac{\partial''}{\partial x_j}\left[\frac{f(x,\theta)}{r^m}\right] = \frac{f_j(x,\theta)}{r^{m+1}}.$$

We require that

$$\frac{\partial' f(x,\theta)}{\partial x_j} \mathrel{\hat{\in}} W_2^{(l-1)}(S), \qquad f_j(x,\theta) \mathrel{\hat{\in}} W_2^{(l-1)}(S)$$

and that these functions, as well as the function $f(x,\theta)$, be continuous on $\Sigma \times S$.

From a)–c) the following properties follow:

c) $f(x',\theta) - f(x,\theta) = \rho^{\varkappa} F(\xi',\xi,\theta)$. Here ξ' and ξ are the images of the points x' and x under stereographic transformation; ρ is the distance between ξ' and ξ; $\varkappa$ is a constant, $0 < \varkappa < 1$, and

$$|F(\xi_1,\eta_1,\theta) - F(\xi_2,\eta_2,\theta)| \le N_1 \left(|\xi_1-\xi_2|^{\sigma} + |\eta_1-\eta_2|^{\sigma}\right),$$

$$N_1 = \text{const}, \qquad 0 < \sigma < 1.$$

d)

$$a(x') - a(x) = \rho^{\varkappa} a(\xi',\xi), \qquad a(\xi',\xi) \in \mathrm{Lip}_{\sigma}(\Sigma).$$

From the conditions listed here and in item 1° it follows that if the function $g(x)$ is orthogonal to all solutions of the homogeneous equation adjoint to equation (1), then the latter has a solution (possibly nonunique) in $L_2(E_m)$. We shall prove that, under the above conditions, any such solution satisfies the condition

$$\left(\frac{1+|x|^2}{1}\right)^{m/2} u(x) \in \mathrm{Lip}_\delta(\Sigma),$$

where δ is determined by the data of the problem.

On the basis of the results of paper ([4]), from conditions b) and c) the following follows: if the characteristic is expanded in a series in spherical functions

$$f(x,\theta) = \sum_{n=1}^{\infty}\sum_{k=1}^{k_n} a_n^{(k)}(x) Y_{n,m}^{(k)}(\theta),$$

then

$$\sum_{n=1}^{\infty}\sum_{k=1}^{k_n} n^{2l}\left|a_n^{(k)}(x)\right|^2 \le C_1, \qquad \sum_{n=1}^{\infty}\sum_{k=1}^{k_n} n^{2l-2}\left|\frac{\partial a_n^{(k)}}{\partial x_j}\right|^2 \le C_1, \qquad C_1 = \mathrm{const}.$$

Let us form the symbol of equation (1):

$$\Phi(x,\theta) = a(x) + \sum_{n=1}^{\infty}\sum_{k=1}^{k_n} \gamma_{n,m} a_n^{(k)}(x) Y_{n,m}^{(k)}(\theta) =$$

$$= a(x) + \int_S f(x,\theta') P(\cos\gamma)\, dS',$$

where ([5])

$$P(\cos\gamma) = \ln|\cos\gamma|^{-1} - \frac{i\pi}{2}\operatorname{sign}\cos\gamma_x$$

and γ is the angle between the radii of the sphere S drawn to the points θ and θ'. Hence

$$\frac{\partial'\Phi}{\partial x_j} = \frac{\partial a}{\partial x_j} + \int_S \frac{\partial' f(x,\theta)}{\partial x_j} P(\cos\gamma)\, dS'.$$

Thus, if $\partial' f(x,\theta)/\partial x_j$ is regarded as the characteristic and $\partial a(x)/\partial x_j$ as the nonintegral coefficient of a certain singular operator, then its symbol will be the function $\partial'\Phi(x,\theta)/\partial x_j$; hence, from the results–

of the results of paper ([4]) it follows that $\partial'\Phi/\partial x_j \in \hat{W}_2^{(l-1+[m/2])}(S)$. The same is true also for the function $\partial'\Phi^{-1}/\partial x_j = -\Phi^{-2}\partial'\Phi/\partial x_j$, since $\inf|\Phi| > 0$.

Let now

$$\Phi^{-1}(x,\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} b_n^{(k)}(x) Y_{n,m}^{(k)}(\theta).$$

Then, by virtue of the same results of paper ([4]),

$$\sum_{n=0}^{\infty}\sum_{k=1}^{k_n} n^{2(l-1+[m/2])}\left|\frac{\partial b_n^{(k)}}{\partial x_j}\right|^2 \le C_2 = \text{const},$$

and, consequently,

$$\sum_{n=0}^{\infty}\sum_{k=1}^{k_n} n^{2l-3}|\gamma_{n,m}|^{-2}\left|\frac{\partial b_n^{(k)}}{\partial x_j}\right|^2 \le C_3 = \text{const}.$$

Hence, in any case, it follows that $\operatorname{grad}' \psi(x,\theta) \in \hat{W}_2^{(l-2)}(S)$, where $\varphi(x,\theta)$ is the characteristic corresponding to the symbol $\Phi^{-1}(x,\theta)$. At the same time, as again follows from the assertion of paper ([4]), $\operatorname{grad}'' \varphi(x,\theta) \in \hat{W}_2^{(l-2)}(S)$. Since $l-2 \ge m$, then, as can be derived from the theorems of paper ([4]), the series obtained by differentiating the series for the function $\varphi(x,\theta)$ with respect to the Cartesian coordinates of the point θ converge absolutely and uniformly. From the formula for the symbol it follows, by conditions b) and c), that the functions $\Phi(x,\theta)$ and $\partial'\Phi(x,\theta)/\partial x_i$ are continuous on $\Sigma \times S$. But then on $\Sigma \times S$ the functions $\Phi^{-1}(x,\theta)$ and $\partial'\Phi^{-1}(x,\theta)/\partial x_i$ are also continuous, and consequently so are the coefficients $b_n^{(k)}(x)$ and their derivatives with respect to x_j. But then

$$\left|\operatorname{grad}_\xi \varphi(x,\theta)\rho^{-m}\right| \le C_4\rho^{-m-1}, \qquad C_4 = \text{const}. \tag{4}$$

Here $\rho = |\xi - \eta|$, where ξ and η are the images of the points x and y under stereographic transformation. Using the corresponding theorem of Giraud ([3]), one can prove that the singular operator

$$b_0^{(1)}(x)u(x) + \int_{E_m} L(x, x-y)u(y)\,dy, \qquad L(x,x-y) = \frac{\varphi(x,\theta)}{r^m}, \tag{5}$$

maps every function $u(x)$ satisfying the condition

$$\left(\frac{1+|x|^2}{2}\right)^{m/2} u(x) \in \operatorname{Lip}_\alpha(\Sigma), \tag{6}$$

into a function satisfying the same condition.

3°. Apply to both sides of equation (1) the operator (5), whose symbol is equal to $\Phi^{-1}(x,\theta)$. This will lead us to an equation of Riesz-Schauder type

$$u(x) + Tu = F(x), \tag{7}$$

which is satisfied by all solutions of equation (1). From what was said above it follows that $F(x)$ satisfies condition (6).

In equation (7) put

$$\left(\frac{1+|x|^2}{2}\right)^{m/2} u(x) = \tilde{u}(\xi), \qquad \left(\frac{1+|x|^2}{2}\right)^{m/2} = \tilde{F}(\xi), \qquad \left(\frac{1+|x|^2}{2}\right)^{m/2} Tu = \tilde{T}\tilde{u};$$

then this equation becomes the following:

$$\tilde{u}(\xi) + \widetilde{T}\tilde{u} = \widetilde{F}(\xi), \tag{8}$$

in which the operator $\widetilde{T}$ is completely continuous in $L_2(\Sigma)$.

It can be proved that

$$\widetilde{T}\tilde{u} = \int_\Sigma \left\{ [a(y) - a(x)] \frac{\varphi(x,\theta)}{\rho^m} + \int_\Sigma \frac{[f(z,\theta_{yz}) - f(x,\theta_{yz})]\varphi(x,\theta_{xz})}{|\xi-\eta|^m |\eta-\zeta|^m} \, d\Sigma_\zeta \right\} \tilde{u}(\eta)\, d\Sigma_\eta. \tag{9}$$

Here ζ is the image of the point z under stereographic transformation,

$$\theta_{yz} = \frac{z-y}{|z-y|}, \qquad \theta_{xz} = \frac{z-x}{|z-x|}.$$

From conditions) and) and from the results of Giraud [(3)] it follows that the kernel of the operator (9) satisfies the conditions of the lemma of the present note. In particular, equation (8) is an integral equation with a weak singularity. The function $\widetilde{F}(\xi) \in \mathrm{Lip}_\alpha(\Sigma)$, and hence is bounded; then every solution of equation (8) belonging to $L_2(\Sigma)$ is bounded. By the lemma, $\widetilde{T}\tilde{u} \in \mathrm{Lip}_\beta(\Sigma)$, where β is determined by the conditions of the problem, and therefore

$$\tilde{u} = (\widetilde{F} - \widetilde{T}\tilde{u}) \in \mathrm{Lip}_\delta(\Sigma),$$

$\delta = \min(\alpha, \beta)$.

4°. The results also extend to the case when the equation has the form

$$a(\xi)u(\xi) + \int_\Gamma K(\xi,\eta)u(\eta)\, d\Gamma_\eta = g(\xi), \qquad g(\xi) \in \mathrm{Lip}_\alpha(\Gamma),$$

where Γ is a closed sufficiently smooth manifold of dimension m, and the kernel is subject, for example, to the following condition: if some neighborhood of the point $\xi \in \Gamma$ is mapped sufficiently smoothly onto some domain of the space E_m, and if x and y are the images of the points ξ and η under this transformation, then

$$K(\xi, \eta) = \frac{f(x, \theta)}{r^m} + \frac{f_0(x, y)}{r^{m-\lambda}},$$

where the singular kernel $r^{-m} f(x, \theta)$ satisfies all the conditions listed above, the exponent $\lambda > 0$, and the function $f_0(x, y)$ is continuously differentiable with respect to the coordinates of both points x and y.

The extension of the results to systems of singular equations is obvious.

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Note: Figure translations are in progress. See original paper for figures.