EQUATIONS OF RELATIVISTIC RADIATION HYDRODYNAMICS

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Abstract

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EQUATIONS OF RELATIVISTIC RADIATION HYDRODYNAMICS

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The main discrepancies in the writing of the equations of hydrodynamics with allowance for radiation are due to the fact that in some works no distinction was made between the definition of the parameters of the radiation field in a fixed and in a moving coordinate system. The discrepancies concerned terms containing the speed of light in the denominator. A complete clarification of which terms should be added to the equations of hydrodynamics in order to take into account the interaction of radiation with a moving medium is possible only within the framework of relativistic theory. In published works, without taking gravitation into account, the case of a linear approximation in v/c has been considered ([1,2]), as has the case of a small deviation of the radiation from equilibrium satisfying Kirchhoff' s law ([3,4]), the case of one-dimensional gas motion ([2]), and, in the general theory of relativity, the case of a photon gas satisfying the Stefan-Boltzmann law and the nonrelativistic Schwarzschild equilibrium condition ([5]). In the present work the radiation field of a moving gas is considered within the framework of the special theory of relativity without the indicated restrictions.

1. Let the radiation field of a medium at rest relative to the reference frame $K^*(x_1^*, x_2^*, x_3^*, t^*)$ be described by the intensity $I^*_{\nu^*}(\mathbf{r}^*, t^*, \mathbf{l}^*)$ (ν^*–frequency; $\mathbf{r}^*(x_1^*, x_2^*, x_3^*)$–coordinates; t^*–time; $\mathbf{l}^*(l_1^*, l_2^*, l_3^*)$–unit vector along the ray) or by the function $n^*_{\nu^*}(\mathbf{r}^*, t^*, \mathbf{l}^*) = I^*_{\nu^*}/h{\nu^*}^3$ of the distribution of photons over directions and frequencies (energies); all quantities with asterisks are taken in the proper reference frame K^*. Through the radiation intensity are expressed: the radiation flux $H^*_{\nu^*}$, the radiation energy density $\varepsilon^*_{\nu^*}$, the components $\pi^*_{\nu^*\alpha\beta}$ of the tensor Π^*_ν of radiation pressure, referred to a unit spectral interval, as well as the corresponding integral characteristics of the radiation. Let, relative to the laboratory reference frame $K(x_1, x_2, x_3, t)$, the volume under consideration move with velocity $\mathbf{v}$. In the reference frame K one can, in exactly the same way as in the frame K^*, define all characteristics of the radiation field. If, instead of n_ν or I_ν, one introduces the Lorentz-invariant function N_ν of the distribution of photons in the 8-dimensional phase space ([6]), $n_\nu = h^3\nu^2 N_\nu/c^2$ (h–Planck' s constant, c –speed of light), then as a result of Lorentz transformations of space-time coordinates we obtain the following relations between the radiation

characteristics in the laboratory and proper reference frames (here $\theta^{-2} = 1-\beta^2$, $\beta = \mathbf{v}/c$, $S_\nu = H_\nu/c^2$):

$$n_\nu = n^*_{\nu^*} L^2, \qquad I_\nu = I^*_{\nu^*} L^3, \qquad I = I^* L^4, \qquad L \equiv \theta\,(1 + l^*_\alpha \beta_\alpha),$$

$$\varepsilon_\nu\, d\nu = \left(\varepsilon^*_{\nu^*} + 2v_\alpha S^*_{\nu^*\alpha} - \pi^*_{\nu^*\chi\lambda}\beta_\chi\beta_\lambda\right)\theta^2 d\nu^*,$$

$$H_{\nu\alpha}\, d\nu = \left[H^*_{\nu^*\alpha} + \theta v_\alpha \varepsilon^*_{\nu^*} - \frac{v_\alpha v_\chi}{v^2} H^*_{\nu^*\chi}(1-\theta-\theta\beta^2) - v_\chi \pi^*_{\nu^*\alpha\chi} + (1-\theta)v_\alpha \frac{v_\chi v_\lambda}{v^2}\pi^*_{\nu^*\chi\lambda}\right]\theta\, d\nu^*, \tag{1}$$

$$\begin{aligned}\pi_{\nu\alpha\beta}\, d\nu = \Big[\pi^*_{\nu^*\alpha\beta} &+ (\theta-1)\frac{v_\chi}{v^2}\left(v_\alpha \pi^*_{\nu^*\chi\beta} + v_\beta \pi^*_{\nu^*\alpha\chi}\right) + \\ &+ 2(1-\theta)\frac{v_\alpha v_\beta}{v^2} S^*_{\nu^*\chi} v_\chi - \theta\left(v_\alpha S^*_{\nu^*\beta} + v_\beta S^*_{\nu^*\alpha}\right) + \\ &+ (1-\theta)^2 \frac{v_\alpha v_\beta}{v^2}\frac{v_\chi v_\lambda}{v^2}\pi^*_{\nu^*\chi\lambda} - \theta^2 \beta_\alpha \beta_\beta \varepsilon^*_{\nu^*}\Big]\, d\nu^*. \end{aligned} \tag{1}$$

The tensor of integral radiative pressure is represented as the sum

$$\begin{aligned}\pi_{\alpha\beta} &= \pi^*_{\alpha\beta} - Q^{*(1)}_{\alpha\beta} + Q^{*(2)}_{\alpha\beta} + Q^{*(3)}_{\alpha\beta},\\ Q^{*(1)}_{\alpha\beta} &= v_\alpha S^*_\beta + v_\beta S^*_\alpha,\\ Q^{*(2)}_{\alpha\beta} &= (\theta-1)\left(v_\alpha \pi^*_{\beta\chi} + v_\beta \pi^*_{\alpha\chi}\right)\frac{v_\chi}{v^2},\\ Q^{*(3)}_{\alpha\beta} &= \left[\frac{2(1-\theta)\theta}{v^2} S^*_\chi v_\chi - \frac{\theta^2}{c^2}\varepsilon^* + \frac{(1-\theta)^2}{v^2}\frac{(\Pi^*\mathbf{v})\mathbf{v}}{v^2}\right][\mathbf{v}\cdot\mathbf{v}],\end{aligned} \tag{2}$$

where $[\mathbf{v}\cdot\mathbf{v}]$ is a dyad. The tensor $Q^{*(1)}_{\alpha\beta}$ was obtained by another method by Kippenhahn ([1]).

2. The equation of radiative transfer has the form

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \frac{\partial I_\nu}{\partial s} = \\ = \rho\eta_\nu + \int_0^\infty \frac{\rho\sigma_{\nu'}}{4\pi}\, d\nu' \int_{4\pi} I_{\nu'}(\mathbf{r},t,\mathbf{l}')\Omega(\nu',\nu,\mathbf{r},t,\mathbf{l}',\mathbf{l})\, d\omega' - \rho(\alpha_\nu + \sigma_\nu) I_\nu(\mathbf{r},t,\mathbf{l}), \tag{3}$$

where ds is the differential along the direction of the ray; $\rho\eta_\nu$, $\rho\alpha_\nu$, $\rho\sigma_\nu$ are the volume coefficients of emission, absorption, and scattering; Ω is the redistribution function. This equation, in Lorentz-invariant form, is rewritten as follows:

$$m_a \frac{\partial N_\nu}{\partial x_a} = -\rho N_\nu (f + f_1) + \rho g + \frac{\rho}{4\pi} \int N_\nu f_1 \Gamma \, d\omega_1, \tag{4}$$

where $d\omega_1 = h^2 \nu \, d\nu \, d\omega / c^4$; m_a is the 4-momentum of the photon gas, with $m_\alpha = h\nu l_\alpha / c^2, \; m_4 = ih\nu/c^2$ (Greek indices are equal to $1, 2, 3$, Latin to $1, 2, 3, 4$); x_α are rectangular coordinates in the Galilean frame of reference; $x_4 = ict$. The functions f, g, f_1, Γ are equal to:

$$f = \frac{h\nu}{c^2} \alpha_\nu, \qquad g = \frac{c^3}{h^3 \nu^2} \eta_\nu, \qquad f_1 = \frac{h\nu}{c^2} \sigma_\nu, \qquad \Gamma = \frac{c^4 \nu'}{h^2 \nu^2} \Omega. \tag{5}$$

They express the physical properties of the gas in a manner invariant with respect to Lorentz transformations and depend on the radiation frequency only in the proper frame of reference. Hence the following Lorentz transformation formulas are obtained:

$$\rho\alpha_\nu = (\rho\alpha_\nu)^* L^{-1}, \qquad \rho\sigma_\nu = (\rho\sigma_\nu)^* L^{-1}, \qquad \rho\eta_\nu = (\rho\eta_\nu)^* L^3,$$
$$\Omega = \Omega^* L^2 (1 + l^*_\alpha \beta_\alpha)^{-1} \theta^{-1}. \tag{6}$$

If redistribution into other frequencies is absent, then instead of the double integral on the right-hand side there will be a single integral over all directions, and instead of the redistribution function—the scattering phase function γ_ν, which transforms like the emission coefficient $\rho\eta_\nu$. The radiative-transfer equation describing the change, in the laboratory frame of reference, of the proper radiation parameters is obtained on the basis of the Lorentz transformations in the following form:

$$\left\{ \frac{1}{c} \frac{\partial}{\partial t} + \left[l^*_\alpha + (\theta - 1) \frac{l^*_\beta v_\beta}{v^2} v_\alpha + \theta \beta_\alpha \right] \frac{\partial}{\partial x_\alpha} \right\} I^*_{\nu^*} \theta^3 (1 + l^*_\alpha \beta_\alpha)^3 =$$
$$= \left\{ (\rho\eta_\nu)^* + \int_0^\infty \frac{(\rho\sigma_{\nu'})^*}{4\pi} \int_{4\pi} I^*_{\nu^*} \Omega^* \, d\nu^* \, d\omega^* - \left[(\rho\alpha_\nu)^* + (\rho\sigma_\nu)^* \right] I^*_{\nu^*} \right\} \theta^2 (1 + l^*_\alpha \beta_\alpha)^2. \tag{7}$$

3. Consider a physical system consisting of an ideal fluid (external forces are absent) and a radiation field (photon gas). The laws of conservation of energy and momentum, as well as of the number of particles, expressed by the equations

$$\frac{\partial T_{rs}}{\partial x_s} = 0, \qquad \frac{\partial (n u^i)}{\partial x^i} = 0 \tag{8}$$

together with the equation for the change in the number of photons—the radiation-transfer equation (7)—and the equations of state, form a system of equations of relativistic hydrodynamics. Here nu^i is the 4-vector of the particle flux, u^i is the 4-velocity, with $u_\alpha = \theta v_\alpha/c$, $u_4 = i\theta$. The energy-momentum tensor T_{rs} is composed of the energy-momentum tensors of the photon gas R_{rs} and of the fluid τ_{rs}, where [6]

$$\tau_{rs} = wu_{ru}s + p\delta_{rs}, \qquad w = e + p, \qquad e = \rho c^2 + \rho U; \tag{9}$$

w, e are the heat function and the internal energy per unit proper volume of the fluid; ρc^2 is the rest energy; ρ is the rest mass density.

We shall write the components of the tensor R_{rs}, proceeding from the physical meaning, in Galilean coordinates of the components of the energy-momentum tensor, in the following form:

$$R_{\alpha\beta} = -\pi_{\alpha\beta}, \qquad R_{\alpha 4} = \frac{i}{c} H_\alpha, \qquad R_{44} = -\varepsilon. \tag{10}$$

The right-hand sides are expressed through the proper radiation characteristics by the formulas given above; from this it follows that the quantities R_{rs} indeed form a covariant tensor. Substituting in (8) the expressions for the components of the tensor $T_{rs} = \tau_{rs} + R_{rs}$, we obtain

$$u_r \frac{\partial (wu^s)}{\partial x^s} + wu^s \frac{\partial u_r}{\partial x^s} + \delta_{rs} \frac{\partial p}{\partial x_s} + \frac{\partial R_{rs}}{\partial x_s} = 0. \tag{11}$$

Hence, together with (8), we have:

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} + \frac{\rho\theta^2}{c^2} v_\alpha \frac{dv_\alpha}{dt} = 0,$$

$$\rho R\theta \frac{dv_\alpha}{dt} + \rho\theta v_\alpha \frac{dR}{dt} + \frac{\partial p}{\partial x_\alpha} - \frac{\partial \pi_{\alpha\beta}}{\partial x_\beta} + \frac{1}{c^2}\frac{\partial H_\alpha}{\partial t} = 0,$$

$$R \equiv \theta\left(1 + \frac{U}{c^2} + \frac{p}{\rho c^2}\right), \tag{12}$$

$$\rho\frac{dU}{dt} - \frac{p}{\rho\theta}\frac{d(\rho\theta)}{dt} + v_\alpha\left(\frac{\partial \pi_{\alpha\beta}}{\partial x_\beta} - \frac{1}{c^2}\frac{\partial H_\alpha}{\partial t}\right) + \frac{1}{\theta^2}\left(\frac{\partial \varepsilon}{\partial t} + \frac{\partial H_\alpha}{\partial x_\alpha}\right) - \rho\theta v^2 \frac{dR}{dt} = 0.$$

For $v/c \ll 1$, retaining terms of order v^2/c^2, we obtain:

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{v} + \frac{\rho}{c^2} v_\alpha \frac{dv_\alpha}{dt} = 0,$$

$$\rho\frac{dv_\alpha}{dt}+\frac{\partial p}{\partial x_\alpha}-\frac{\partial\pi_{\alpha\beta}}{\partial x_\beta}+\frac{1}{c^2}\frac{\partial H_\alpha}{\partial t}+\frac{\rho}{c^2}\frac{d}{dt}\left[\left(\frac{v^2}{2}+U+\frac{p}{\rho}\right)v_\alpha\right]=0; \tag{13}$$

$$\rho\frac{dU}{dt}-\frac{p}{\rho}\frac{d\rho}{dt}+\frac{\partial H_\alpha}{\partial x_\alpha}+\frac{\partial\varepsilon}{\partial t}+v_\alpha\frac{\partial\pi_{\alpha\beta}}{\partial x_\beta}-\frac{v_\alpha}{c^2}\frac{\partial H_\alpha}{\partial t}$$

$$-\frac{v^2}{c^2}\left[\frac{\rho}{2}\frac{dU}{dt}+v_\alpha\frac{\partial p}{\partial x_\alpha}+\rho\left(1-\frac{p}{\rho v^2}\right)v_\alpha\frac{dv_\alpha}{dt}-v_\alpha\frac{\partial\pi_{\alpha\beta}}{\partial x_\beta}+\frac{v_\alpha}{c^2}\frac{\partial H_\alpha}{\partial t}\right]=0.$$

The density of the radiation momentum is equal to $\mathbf{S}=\mathbf{H}/c^2$, and therefore we have retained also the terms containing v^2H/c^4. Here the radiation characteristics are taken in the laboratory frame of reference. For the radiation parameters in the prop–

in the proper reference frame we obtain:

$$\begin{gathered}\rho\frac{d\mathbf{v}}{dt}+\nabla p-\operatorname{Div}\Pi^*+\frac{d\mathbf{S}^*}{dt}+\mathbf{S}^*\operatorname{div}\mathbf{v}+\mathbf{v}\operatorname{div}\mathbf{S}^*+(\mathbf{S}^*\nabla)\mathbf{v}+\\
+\frac{1}{2c^2}\left\{2\frac{d}{dt}\left[\mathbf{v}(\varepsilon^*-\Pi^*)\right]+\mathbf{v}(2\varepsilon^*-\Pi^*)\operatorname{div}\mathbf{v}+(\mathbf{v}\nabla)(\mathbf{v}\Pi^*)-\right.\\
-\left[(\mathbf{v}\Pi^*)\nabla+(\Pi^*\nabla)\mathbf{v}+\mathbf{v}\operatorname{Div}\Pi^*\right]\mathbf{v}+\left[\frac{d\mathbf{S}^*}{dt}+\mathbf{S}^*\operatorname{div}\mathbf{v}+\mathbf{v}\operatorname{div}\mathbf{S}^*+\right.\\
\left.+(\mathbf{S}^*\nabla)\mathbf{v}+\mathbf{v}(\mathbf{S}^*\nabla)+\mathbf{S}^*(\mathbf{v}\nabla)\right]v^2+\left(2\frac{d}{dt}+\frac{\partial}{\partial t}\right)\left[\mathbf{v}(\mathbf{v}\mathbf{S}^*)\right]+\\
\left.+2(\mathbf{v}\mathbf{S}^*)\mathbf{v}\operatorname{div}\mathbf{v}-2\rho\frac{d}{dt}\left[\left(\frac{v^2}{2}+U+\frac{p}{\rho}\right)\mathbf{v}\right]\right\}=0,\end{gathered} \tag{14}$$

$$\begin{gathered}\rho\frac{dU}{dt}-\frac{p}{\rho}\frac{d\rho}{dt}+\operatorname{div}\mathbf{H}^*+\frac{d\varepsilon^*}{dt}+\varepsilon^*\operatorname{div}\mathbf{v}-(\Pi^*\mathbf{v})\mathbf{v}+\\
+\frac{1}{2}\left(\frac{d}{dt}+\frac{\partial}{\partial t}\right)(\mathbf{v}\mathbf{S}^*)+\mathbf{S}^*\frac{d\mathbf{v}}{dt}+\frac{\mathbf{v}}{2}\mathbf{S}^*\operatorname{div}\mathbf{v}-\frac{v^2}{2}\operatorname{div}\mathbf{S}^*+\\
+\frac{\mathbf{v}}{c^2}\left\{\varepsilon^*\frac{d\mathbf{v}}{dt}-\Pi^*\frac{\partial\mathbf{v}}{\partial t}+v^2\operatorname{Div}\Pi^*-\left[\nabla\frac{v^2}{2}+(\mathbf{v}\nabla)\mathbf{v}\right]\Pi^*+\right.\\
+\frac{v^2}{2}\left[\frac{\partial}{\partial t}-3\frac{d}{dt}+\frac{3}{2v^2}\frac{\partial v^2}{\partial t}-\frac{7}{4}\operatorname{div}\mathbf{v}+\frac{5}{4}(\mathbf{v}\nabla)+\frac{5}{4}\frac{\mathbf{v}\nabla}{v^2}v^2\right]\mathbf{S}^*-\\
-\mathbf{v}\left[\frac{\rho}{2}\frac{dU}{dt}+\mathbf{v}\nabla p+\rho\left(1-\frac{p}{\rho v^2}\right)\mathbf{v}\frac{d\mathbf{v}}{dt}+\right.\\
\left.\left.+\frac{\mathbf{S}^*}{2}\left(\nabla v^2-\frac{5}{4}(\mathbf{v}\nabla)\mathbf{v}-\frac{\partial\mathbf{v}}{\partial t}\right)-\frac{9}{8}v^2\operatorname{div}\mathbf{S}^*\right]\right\}=0.\end{gathered}$$

Hence one obtains the equations in the nonrelativistic limiting case for the radiation parameters in the laboratory coordinate system:

$$\rho \frac{d\mathbf{v}}{dt} + \nabla p - \operatorname{Div} \Pi + \frac{\partial \mathbf{S}}{\partial t} = 0,$$

$$\rho \frac{dU}{dt} - \frac{p}{\rho} \frac{d\rho}{dt} + \operatorname{div} \mathbf{H} + \frac{\partial \varepsilon}{\partial t} + \mathbf{v} \operatorname{Div} \Pi - \mathbf{v} \frac{\partial \mathbf{S}}{\partial t} = 0 \tag{15}$$

and for the radiation functions in the proper reference frame

$$\rho \frac{d\mathbf{v}}{dt} + \nabla p - \operatorname{Div} \Pi^* + \frac{d\mathbf{S}^*}{dt} + \mathbf{v} \operatorname{div} \mathbf{S}^* + (\mathbf{S}^* \nabla)\mathbf{v} + \mathbf{S}^* \operatorname{div} \mathbf{v} = 0, \tag{16}$$

$$\rho \frac{dU}{dt} + \frac{d\varepsilon^*}{dt} - \frac{p + \varepsilon^*}{\rho} \frac{d\rho}{dt} + \operatorname{div} \mathbf{H}^* - \frac{\pi^*_{\alpha\beta}}{2} \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha} \right) + \frac{d(\mathbf{v}\mathbf{S}^*)}{dt} + \mathbf{S}^* \frac{d\mathbf{v}}{dt} +$$
$$+ \left[\mathbf{S}^* \operatorname{div} \mathbf{v} - \mathbf{S}^* (\mathbf{v}\nabla) - \mathbf{v} \operatorname{div} \mathbf{S}^* \right] \frac{\mathbf{v}}{2} - \frac{\mathbf{v}}{2} (\mathbf{v}\nabla) \mathbf{S}^* = 0.$$

Equations (15) coincide with those obtained earlier ([2]), as do (16), if the terms containing H/c^2 are discarded ([8]).

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Note: Figure translations are in progress. See original paper for figures.